Article:

Gould, Victoria Ann Rosalind orcid.org/0000-0001-8753-693X and Quinn-Gregson, Thomas David (2019) $\aleph_0$-categoricity of semigroups. Semigroup forum. pp. 260-292. ISSN 1432-2137

Semigroup Forum
https://doi.org/10.1007/s00233-019-10002-7

RESEARCH ARTICLE

$\aleph_0$-categoricity of semigroups

Victoria Gould[1] · Thomas Quinn-Gregson[2]

Received: 19 February 2018 / Accepted: 14 January 2019

Abstract
In this paper we initiate the study of $\aleph_0$-categorical semigroups, where a countable semigroup S is $\aleph_0$-categorical if, for any natural number n, the action of its group of automorphisms $\mathrm{Aut}(S)$ on S^n has only finitely many orbits. We show that $\aleph_0$-categoricity transfers to certain important substructures such as maximal subgroups and principal factors. We examine the relationship between $\aleph_0$-categoricity and a number of semigroup and monoid constructions, namely Brandt semigroups, direct sums, 0-direct unions, semidirect products and $\mathcal{P}$-semigroups. As a corollary, we determine the $\aleph_0$-categoricity of an E-unitary inverse semigroup with finite semilattice of idempotents in terms of that of the maximal group homomorphic image.

Keywords $\aleph_0$-categorical · Semigroups · Semidirect product

1 Introduction

The concept of $\aleph_0$-categoricity is rooted in model theory. Let L be a first-order language and let T be a *theory* in L, that is, T is a set of sentences in L. The theory T is $\aleph_0$-*categorical* if T has exactly one countable model, up to isomorphism. In other words, there is a countable L-structure in which all the sentences of T are true, and is such that it is isomorphic to any other countable L-structure with the same property. We then say that an L-structure A is $\aleph_0$-*categorical* if $\mathrm{Th}(A)$ is $\aleph_0$-categorical, where $\mathrm{Th}(A)$ is the set of all sentences of L which are true in A. Thus, A is $\aleph_0$-categorical if it is determined up to isomorphism by its first-order properties. Morley's celebrated

Communicated by Mark V. Lawson.

✉ Victoria Gould
victoria.gould@york.ac.uk

Thomas Quinn-Gregson
thomas.quinn-gregson@tu-dresden.de

1 Department of Mathematics, University of York, York YO10 5DD, UK

2 Institut für Algebra, Technische Universität Dresden, 01062 Dresden, Germany

Published online: 29 January 2019

categoricity theorem [25] was the impetus for the development of the rich area of model theory known as *stability theory* (see [28]).

It is a natural question to determine the $\aleph_0$-categorical members of any class of relational or algebraic structures: in our case, semigroups and monoids. This may be addressed without recourse to specialist model theory, in view of the following result, independently accredited to Engeler [8], Ryll-Nardzewski [37], and Svenonius [41], but commonly referred to as the Ryll-Nardzewski Theorem (RNT). Although stated in generality, we will apply it almost entirely in the context of semigroups, and semigroups with augmented structure (such as an identity, a partial order, or distinguished subsets).

Theorem 1.1 (Ryll-Nardzewski Theorem) *A countable structure A is $\aleph_0$-categorical if and only if for each $n \in \mathbb{N}$ the natural action of* $\mathrm{Aut}(A)$ *on A^n has only finitely many orbits.*

A number of authors have considered $\aleph_0$-categoricity for algebraic structures. Rosenstein [34] classified $\aleph_0$-categorical abelian groups, and an extensive overview of the results for groups is given in [1]. Baldwin and Rose [2] investigated $\aleph_0$-categoricity for rings. For semigroups per se, little is known in this context. This paper and a sequel [29] provide an introduction to the study of $\aleph_0$-categorical semigroups.

The notion of $\aleph_0$-categoricity has strong connections with that of homogeneity, where a structure is *homogeneous* if every isomorphism between finitely generated substructures extends to an automorphism. A homogeneous structure is $\aleph_0$-categorical if, for each $n \in \mathbb{N}$, there is a finite bound on the number of non-isomorphic substructures which can be generated by n elements [18]. As we explain in Sect. 2 this leads to a wide class of $\aleph_0$-categorical semigroups arising from the second author's work on the homogeneity of semigroups and, in particular, of semigroups of idempotents (bands) [30–32]. However, certainly not every $\aleph_0$-categorical semigroup is homogeneous, as shown in Sect. 2.

There are two main approaches to the study of $\aleph_0$-categoricity. The first half of this paper will be in line with the 'preservation theorems' approach. In particular, we shall investigate when the $\aleph_0$-categoricity of a semigroup passes to subsemigroups, quotients and certain direct sums. This is certainly a popular path to take: we mention here Grzegorczyk's handy result that $\aleph_0$-categoricity (of a general structure) is preserved by finite direct products [16]. In [42], Waszkiewicz and Weglorz showed that $\aleph_0$-categoricity of a structure is preserved by Boolean extensions by $\aleph_0$-categorical Boolean algebras, a result later generalized by Schmerl [39] to filtered Boolean extensions. In [38] Sabbagh proved that the group $\mathrm{GL}_n(R)$ of invertible $n \times n$ matrices over an $\aleph_0$-categorical ring R inherits $\aleph_0$-categoricity; the corresponding result for the semigroup $M_n(R)$ of all $n \times n$ matrices follows easily from Theorem 1.1. The final two sections fit into the 'classification' approach: determining the $\aleph_0$-categoricity of semigroups in certain classes built from $\aleph_0$-categorical components. In particular, we classify $\aleph_0$-categorical Brandt semigroups, 0-direct unions and certain $\aleph_0$-categorical semidirect products, including the case where the semigroup being acted upon is a finite semilattice.

A number of known classifications will be of use in our work, including the $\aleph_0$-categoricity of linear orders [33], which serve as examples of $\aleph_0$-categorical semilattices. For algebraic structures, the difficulty in achieving full classifications has long been apparent. Algebras in general do not display the high degree of global symmetry required for $\aleph_0$-categoricity, and knowledge of the group of automorphisms $\mathrm{Aut}(A)$ of an algebra A is not always important in determining A. However, significant results are available for groups and rings, largely because the underlying group structure forces a degree of symmetry. The former are of particular importance to this paper, since maximal subgroups of $\aleph_0$-categorical semigroups are $\aleph_0$-categorical (see Corollary 3.7). Our work on $\aleph_0$-categoricity for semigroups shows that even if we know that various constituent parts of a semigroup S, such as the maximal subgroups, are $\aleph_0$-categorical, it can be hard to determine when S itself is $\aleph_0$-categorical, due to the complex way in which the constituents are glued together.

Throughout the paper we develop tools for ascertaining $\aleph_0$-categoricity of semigroups, built on Theorem 1.1 with increasing degrees of complexity, which are made use of as follows. In Sect. 2 we show that any $\aleph_0$-categorical semigroup is periodic with bounded index and period (Corollary 2.3), and it can be cut into $\aleph_0$-categorical 'slices' which satisfy the additional property of being *characteristically [0-]simple* (Proposition 2.18). We also prove the existence of an $\aleph_0$-categorical nil semigroup that is not nilpotent, a situation that contrasts to that in ring theory [4] (Theorem 2.16). In Sect. 3 we develop a notion for subsemigroups of being *relatively characteristic*, which is somewhat weaker than the standard notion of being characteristic (i.e. preserved by all automorphisms). We are then able to demonstrate how $\aleph_0$-categoricity is inherited by maximal subgroups (Corollary 3.7), principal factors (Theorem 3.12), and certain other quotients (Corollary 3.11). We begin Sect. 4 by showing that a Brandt semigroup $\mathcal{B}^0[G; I]$ is $\aleph_0$-categorical if and only if G is an $\aleph_0$-categorical group (Theorem 4.2), a result which will be extended in the subsequent paper [29]. We also consider direct sums and 0-direct sums, determining when a direct sum of finite monoids or semigroups is $\aleph_0$-categorical (Theorem 4.4 and Proposition 4.5, respectively) and, via an analysis of 0-direct sums, when a primitive inverse semigroup is $\aleph_0$-categorical (Corollary 4.10). Finally in Sect. 5 we examine how $\aleph_0$-categoricity interacts with semidirect products, and with the construction of McAlister $\mathcal{P}$-semigroups $\mathcal{P} = \mathcal{P}(G, \mathcal{X}, \mathcal{Y})$ in terms of the $\aleph_0$-categoricity of G, $\mathcal{X}$ and $\mathcal{Y}$. In particular we show that if $\mathcal{Y}$ is finite then the $\aleph_0$-categoricity of $\mathcal{P}$ depends only on G (Theorem 5.12).

We denote the set of natural numbers (without 0) by $\mathbb{N}$ and we write $\mathbb{N}^0$ to mean $\mathbb{N} \cup \{0\}$. Given a semigroup S and $n \in \mathbb{N}$, we let S^n denote both the set of n-tuples of S and all products of length n of elements from S; the meaning of the notation S^n should be clear from the context. Occasionally we make use of the convention that $S^0 = \emptyset$ denotes the set of 0-tuples of elements of S. The set of idempotents of a semigroup S will be denoted $E(S)$. We take the convention that $\emptyset$ is always an ideal of a semigroup. The identity bijection of a set X will be denoted by I_X.

This article does not require any background in model theory, but we refer the reader to [18] for an introductory study, and to [9] for an overview of $\aleph_0$-categorical structures.

2 First examples of $\aleph_0$-categorical semigroups

As commented in the Introduction, our main tool in determining $\aleph_0$-categoricity is Theorem 1.1. An immediate consequence worth highlighting is:

Corollary 2.1 *Finite semigroups are $\aleph_0$-categorical.*

Recall from [18] that a structure M is *uniformly locally finite* (ULF) if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for every substructure T of M, if T has a generating set of cardinality at most n, then T has cardinality at most $f(n)$. Rosenstein [34, Theorem 16] showed that an $\aleph_0$-categorical group is ULF, a result later proved for general structures:

Proposition 2.2 [18, Corollary 7.3.2] *An $\aleph_0$-categorical structure is ULF.*

We show below, by taking the example of an ω-chain, that the converse to Proposition 2.2 need not hold. However, from the fact that an $\aleph_0$-categorical semigroup is ULF, we may immediately deduce the following:

Corollary 2.3 *An $\aleph_0$-categorical semigroup S is periodic, with bounded index and period. Consequently, $E(S) \neq \emptyset$ and $\mathcal{D} = \mathcal{J}$.*

A converse to Proposition 2.2 holds if we restrict our attention to homogeneous structures, where a structure M is *homogeneous* if every isomorphism between finitely generated substructures of M extends to an automorphism of M.

Proposition 2.4 [18, Corollary 7.4.2] *A homogeneous ULF structure is $\aleph_0$-categorical.*

Since McLean showed [24] that any band is ULF, we immediately have:

Corollary 2.5 *Homogeneous bands and homogeneous semilattices are $\aleph_0$-categorical.*

The homogeneity of both bands and inverse semigroups was studied by the second author, with results appearing in [31] (where a complete characterisation of homogeneous bands is given) and [32], respectively; see also the thesis [30] of the second author. Complete characterisations of homogeneous semilattices appear in [6,7], and of $\aleph_0$-categorical linear orders in [33]. Let $\mathcal{Q} = \mathbb{Q} \cup \{\infty\}$ be linearly ordered by extending the usual order in $\mathbb{Q}$ by adjoining a maximum element ∞. It follows from these papers that $\mathcal{Q}$ is not a homogeneous semilattice but is an $\aleph_0$-categorical semilattice.

Further, an ω-chain

$$x_1 < x_2 < \dots$$

or an inverse ω-chain

$$x_1 > x_2 > \dots$$

are not $\aleph_0$-categorical. This contrasts starkly with the fact that an abelian group is $\aleph_0$-categorical if and only if it is periodic of bounded index [34].

At this stage it is convenient to fix some notation to help in the implementation of the RNT. Let A, B be sets, $n \in \mathbb{N}^0$, $\phi : A \to B$ a map, $\overline{a} = (a_1, \dots, a_n)$ an

n-tuple of elements of A and $M \subseteq A$. Then $\overline{a}\phi$ denotes the n-tuple of B given by $(a_1\phi, \ldots, a_n\phi)$, and $M\phi$ denotes the subset $\{m\phi : m \in M\}$ of B.

Given a semigroup S and pair $\overline{a} = (a_1, \ldots, a_n)$, $\overline{b} = (b_1, \ldots, b_n)$ of n-tuples of S, then we say that $\overline{a}$ is *automorphically equivalent to/has the same n-automorphism type as* $\overline{b}$ (in S) if there exists an automorphism ϕ of S such that $\overline{a}\phi = \overline{b}$. We denote this relation on S^n by $\overline{a} \sim_{S,n} \overline{b}$, using the same notation for the restriction to subsets of S^n. Hence, by the RNT, to prove that S is $\aleph_0$-categorical it suffices to show that, for each n, there exists a finite list of elements of S^n such that every element of S^n is automorphically-equivalent to an element of the list; equivalently, in any countably infinite list of elements of S^n, we can find two distinct members that are automorphically-equivalent. We augment our notation as follows. Suppose that $\overline{X}$ is a finite tuple of elements of S. Let $\mathrm{Aut}(S; \overline{X})$ denote the subgroup of $\mathrm{Aut}(S)$ consisting of those automorphisms which fix $\overline{X}$. We say that S is $\aleph_0$-*categorical over* $\overline{X}$ if $\mathrm{Aut}(S; \overline{X})$ has only finitely many orbits in its action on S^n for each $n \in \mathbb{N}$. We denote the resulting equivalence relation on S^n as $\sim_{S,\overline{X},n}$, and a pair of $\sim_{S,\overline{X},n}$-equivalent n-tuples are said to be *automorphically equivalent over* $\overline{X}$.

Lemma 2.6 is a simple generalisation of [18, Exercise 7.3.1], and follows immediately from the RNT.

Lemma 2.6 *Let S be a semigroup and $\overline{X}$ a finite tuple of elements of S. For any subset T of S, we have that $|T^n/\sim_{S,n}|$ is finite for all $n \in \mathbb{N}$ if and only if $|T^n/\sim_{S,\overline{X},n}|$ is finite for all $n \in \mathbb{N}$. In particular, S is $\aleph_0$-categorical if and only if S is $\aleph_0$-categorical over $\overline{X}$.*

Example 2.7 Consider the countably infinite null semigroup N, with multiplication $xy = 0$ for all $x, y \in N$. Since any permutation of the non-zero elements gives an isomorphism, it is clear that N is homogeneous. Clearly N is ULF, so that it is also $\aleph_0$-categorical. Indeed, N provides a good illustration of the RNT. A pair of n-tuples $\overline{a} = (a_1, \ldots, a_n)$ and $\overline{b} = (b_1, \ldots, b_n)$ are automorphically equivalent if and only if the positions (if any exist) of the entries equal to 0 are the same in $\overline{a}$ and $\overline{b}$, and for any $1 \leq i, j \leq n$ we have $a_i = a_j$ if and only if $b_i = b_j$. Since there are finitely many choices for each of these conditions, it follows that N is $\aleph_0$-categorical by the RNT.

It is worth formalising the points raised in the above argument, as they will be used throughout this paper. They are based on the following lemma, which may be proven by a simple counting argument.

Lemma 2.8 *Let X be a set and $\gamma_1, \ldots, \gamma_r$ be a finite list of equivalence relations on X with $\gamma_1 \cap \gamma_2 \cap \cdots \cap \gamma_r$ contained in an equivalence relation σ on X. Then*

$$|X/\sigma| \leq \prod_{1 \leq i \leq r} |X/\gamma_i|.$$

We use the RNT in conjunction with Lemma 2.8 to prove that a semigroup S is $\aleph_0$-categorical in the following way.

Corollary 2.9 *Let S be a semigroup and for each $n \in \mathbb{N}$, let $\gamma_1, \ldots, \gamma_{r(n)}$ be a finite list of equivalence relations on S^n such that S^n/γ_i is finite for each $1 \leq i \leq r(n)$ and*

$$\gamma_1 \cap \gamma_2 \cap \cdots \cap \gamma_{r(n)} \subseteq \sim_{S,n}.$$

Then S is $\aleph_0$-categorical.

The equivalence relations γ_i appearing in Corollary 2.9 often arise from first considering a specific condition on a set (for example, the condition on $(\mathbb{N}^0)^n$ for some $n \in \mathbb{N}$ that two n-tuples have any 0 entries in the same positions), and then by analysing the number of different ways in which the condition may be fulfilled (for example, considering the subsets of $\{1, \ldots, n\}$ corresponding to 0 entries; see Example 2.10 (i) below). Where confusion is unlikely to arise, we may refer such equivalences in a less formal way, as follows. Suppose that we have an equivalence relation σ on S^n that arises from different ways in which a given condition may be fulfilled; if S^n/σ is finite, then we say the condition has *finitely many choices*.

Example 2.10 (i) If S is a semigroup with zero (as in Example 2.7), the equivalence $\sim_0$ on S^n defined by the rule that

$$(a_1, \ldots, a_n) \sim_0 (b_1, \ldots, b_n) \Leftrightarrow \{i : a_i = 0\} = \{i : b_i = 0\}$$

corresponds to the condition on n-tuples that they have the non-zero entries in the same positions, and this condition has 2^n choices.

(ii) In a similar way to that of Example 2.7, given a set X, we may impose a condition on a pair of n-tuples of X which states that if a pair of entries in one of the tuples are equal then the same is true for the other tuple. Note that here we do not need to concern ourselves with 0 entries. Formally, we define an equivalence $\natural_{X,n}$ on X^n by

$$(a_1, \ldots, a_n) \natural_{X,n} (b_1, \ldots, b_n) \text{ if and only if } [a_i = a_j \Leftrightarrow b_i = b_j, \text{ for each } i, j]. \tag{2.1}$$

It is clear that a pair of n-tuples $\overline{a}$ and $\overline{b}$ are $\natural_{X,n}$-equivalent if and only if there exists a bijection $\phi : \{a_1, \ldots, a_n\} \to \{b_1, \ldots, b_n\}$ such that $a_i\phi = b_i$. Moreover, the number of $\natural_{X,n}$-classes of X^n is equal to the number of ways of partitioning a set of size n, which is called the *nth Bell number*, denoted B_n (for a formulation, see [36]). In particular B_n is finite, for each $n \in \mathbb{N}$. Note also that if S is a semigroup then

$$\overline{a} \sim_{S,n} \overline{b} \Rightarrow \overline{a} \natural_{S,n} \overline{b}.$$

We will see in this paper that $\aleph_0$-categoricity 'works well' in conjunction with fixing finite sets of elements within semigroups. For example, in order to prove that S has finitely many n-automorphism types, it suffices to consider n-tuples of $S \backslash T$, where T is finite:

Proposition 2.11 *Let S be a semigroup and T a finite subset of S. Then S is $\aleph_0$-categorical if and only if $|(S \backslash T)^n / \sim_{S,n}|$ is finite for each $n \in \mathbb{N}$.*

Proof If S is $\aleph_0$-categorical then $|S^n/\sim_{S,n}|$ is finite by the RNT, and thus so is $|(S\backslash T)^n/\sim_{S,n}|$.

For the converse, we begin by fixing some notation. Let A be a subset of S and $\overline{s}=(s_1,\ldots,s_n)$ an n-tuple of S. Then we let

$$\overline{s}[A]:=\{k\in\{1,\ldots,n\}:s_k\in A\}$$

be the set of positions of entries of $\overline{s}$ which lie in A. If $\overline{s}[A]=\{k_1,\ldots,k_r\}$ is such that $k_1<k_2<\cdots<k_r$ then we obtain an r-tuple of A given by

$$\overline{s}^A:=(s_{k_1},\ldots,s_{k_r}).$$

Let $T=\{t_1,\ldots,t_r\}$ and take $\overline{T}=(t_1,\ldots,t_r)\in S^r$. Let $\overline{a}$ and $\overline{b}$ be n-tuples of S under the conditions that

(1) $\overline{a}[T]=\overline{b}[T]$ with $\overline{a}^T=\overline{b}^T$,
(2) $\overline{a}^{S\backslash T}$ and $\overline{b}^{S\backslash T}$ are automorphically equivalent over $\overline{T}$.

Condition (1) has $(|T|+1)^n$ possibilities, which is finite since T is. Each $|(S\backslash T)^m/\sim_{S,m}|$ is finite by our hypothesis, and so $|(S\backslash T)^m/\sim_{S,\overline{X},m}|$ is also finite for each $m\in\mathbb{N}$ by Lemma 2.6. Hence condition (2) has finitely many choices, and the total number of choices is therefore finite. By condition (2) there exists $\phi\in\mathrm{Aut}(S;\overline{T})$ with $\overline{a}^{S\backslash T}\phi=\overline{b}^{S\backslash T}$. Since $t_i\phi=t_i$ for each $1\le i\le r$ we have $\overline{a}^T\phi=\overline{a}^T=\overline{b}^T$, and it follows that $\overline{a}\phi=\overline{b}$. The result is then immediate from Lemma 2.8. □

For a semigroup S we let $S^{\underline{1}}$ [S^0] denote S with an identity [zero] adjoined (whether or not S already has such an element). We note that, by convention, S^1 denotes S with *an identity adjoined if necessary*. The next result follows from Proposition 2.11 and the fact that automorphisms of $S^{\underline{1}}$ and S^0 are exactly extensions of automorphisms of S.

Corollary 2.12 *The following are equivalent for any semigroup S:*

(1) S is $\aleph_0$-categorical;
(2) S^0 is $\aleph_0$-categorical;
(3) $S^{\underline{1}}$ is $\aleph_0$-categorical.

In a similar fashion to that in Corollary 2.12 we can build new $\aleph_0$-categorical semigroups from given ingredients, provided the ingredients interact in a relatively simplistic way (see Proposition 2.20 below).

We now introduce an important notion for $\aleph_0$-categorical semigroups.

Definition 2.13 A subset [subsemigroup, ideal] A of a semigroup S is *characteristic* if it is invariant under automorphisms of S; that is, $A\phi=A$ for all $\phi\in\mathrm{Aut}(S)$.

Clearly any subset A of a semigroup S is characteristic if and only if it is a union of $\sim_{S,1}$-classes, and if A is a characteristic subset of S then $\langle A\rangle$ is a characteristic subsemigroup of S.

Let S be a semigroup with zero 0, and let $n\in\mathbb{N}$. We say that S is *nil* of degree n if for all $a\in S$ we have $a^n=0$, and S is *nilpotent* of degree n if $S^n=0$. If R is a ring, then we say R is *nil/nilpotent* (*of degree* n) if its underlying multiplicative semigroup is nil/nilpotent (of degree n).

Corollary 2.14 *Let S be $\aleph_0$-categorical. Then*

(1) there are finitely many characteristic subsets of S;
(2) any characteristic subsemigroup of S is $\aleph_0$-categorical;
(3) any ideal S^m is characteristic and hence $\aleph_0$-categorical;
(4) for some $n \in \mathbb{N}$ we have $S^n = S^{n+1}$, so that $S^n = S^m$ for all $m \geq n$;
(5) with n as in (4), for any $k < \ell \leq n$ we have that S^ℓ is an ideal of S^k and the Rees quotient S^k/S^ℓ is $\aleph_0$-categorical;
(6) with n as in (4), S/S^n is $\aleph_0$-categorical and nilpotent of degree n.

Proof (1) follows from the fact that a subset is characteristic if and only if it is a union of $\sim_{S,1}$ classes and (2) is immediate from the definition of characteristic subsemigroup. (3) is clear and then (4) is immediate from (1), (3), and the fact that $S^n \supseteq S^{n+1}$.

For (5), observe that S^ℓ is an ideal of S^k. To see that S^k/S^ℓ is $\aleph_0$-categorical, let $m \in \mathbb{N}$ and consider a list of m-tuples of elements of S^k/S^ℓ. By Proposition 2.11 we may assume all of these elements are non-zero, and we may thus identify them with elements of S^k. Since S^k is $\aleph_0$-categorical we may find a distinct pair $(a_1, \ldots, a_m)$ and $(b_1, \ldots, b_m)$ in our list and $\phi \in \mathrm{Aut}(S^k)$ such that $a_i\phi = b_i$ for $1 \leq i \leq m$. It is easy to see that ϕ induces an automorphism ϕ' of S^k/S^ℓ, and regarded as m-tuples of S^k/S^ℓ, we have $(a_1, \ldots, a_m)\phi' = (b_1, \ldots, b_m)$. Hence by Proposition 2.11, S^k/S^ℓ is $\aleph_0$-categorical.

For (6), observe S/S^n is nilpotent of degree n. The rest of the statement follows from (5). $\square$

Corollary 2.14 shows that any $\aleph_0$-categorical semigroup is associated with a nilpotent one. A major result for $\aleph_0$-categorical rings states that any $\aleph_0$-categorical nil ring of degree n is nilpotent of degree n [4]. We show that the corresponding result is not true for semigroups, by constructing a countably infinite $\aleph_0$-categorical commutative semigroup S such that S is nil of degree 2 and $S = S^2$, so that certainly S is non-nilpotent.

A commutative semigroup S, nil of degree 2 such that $S = S^2$ is called a *zs-semigroup*. Some progress has been made in understanding the structure of zs-semigroups, including [20] and [12]. In [12], a simple example of a zs-semigroup is constructed, which is very similar to that given below. However, we need to start with a countable atomless Boolean algebra in order to ensure the resulting zs-semigroup is $\aleph_0$-categorical.

Remark 2.15 There exists a unique, up to isomorphism, countable atomless Boolean algebra B [13, Theorem 10].[1] Since atomless Boolean algebras are axiomatisable, it follows that B is $\aleph_0$-categorical. We can construct B in a number of ways, including the Lindenbaum algebra of propositional logic [21, Chapter 6]. For our purposes it is convenient to use the construction given in [15, Corollary 23] via certain subsets of $[0, 1]$. Let B be the set of all subsets of $[0, 1] \cap \mathbb{Q}$ of the form

$$(a_0] \cup (b_1, a_1] \cup \cdots \cup (b_{n-1}, a_{n-1}]$$

[1] We thank Prof. John Truss of the University of Leeds for bringing this example to our attention.

where $a_0, b_1, a_1, \ldots, a_{n-1} \in \mathbb{Q}$ with $0 < a_0 < b_1 < a_1 < \cdots < b_{n-1} < a_{n-1} \leq 1$ and $(a_0]$ may be missing. It is clear that B forms a subalgebra of the Boolean algebra of subsets of $(0, 1] \cap \mathbb{Q}$ and is atomless. Moreover, if $A = (a_0] \cup (b_1, a_1] \cup \cdots \cup (b_{n-1}, a_{n-1}]$ is a non-empty element of B then, taking any $x \in (b_1, a_1) \cap \mathbb{Q}$, we have that

$$A = \big((a_0] \cup (b_1, x]\big) \cup \big((x, a_1] \cup \cdots \cup (b_{n-1}, a_{n-1}]\big)$$

and

$$\big((a_0] \cup (b_1, x]\big) \cap \big((x, a_1] \cup \cdots \cup (b_{n-1}, a_{n-1}]\big) = \emptyset.$$

Theorem 2.16 *There is an $\aleph_0$-categorical non-nilpotent, nil semigroup.*

Proof We show there is an $\aleph_0$-categorical zs-semigroup that is not nilpotent.

Let $B = (B, \wedge, \vee, 0, 1)$ be the countable atomless Boolean algebra and let $B^* = B \backslash \{0\}$. Define an operation $+$ on B^* by

$$A + A' = \begin{cases} A \vee A' & \text{if } A \wedge A' = 0, \\ 1 & \text{else.} \end{cases}$$

Since $\vee$ is an associative operation and $\wedge$ distributes over $\vee$, we have for any $A_1, A_2, A_3 \in B^*$,

$$\begin{aligned}
(A_1 + A_2) + A_3 &= \begin{cases} (A_1 \vee A_2) + A_3 & \text{if } A_1 \wedge A_2 = 0, \\ 1 + A_3 & \text{else,} \end{cases} \\
&= \begin{cases} (A_1 \vee A_2) \vee A_3 & \text{if } A_1 \wedge A_2 = 0 \text{ and } (A_1 \vee A_2) \wedge A_3 = 0, \\ 1 & \text{else,} \end{cases} \\
&= \begin{cases} A_1 \vee (A_2 \vee A_3) & \text{if } A_1 \wedge A_2 = 0 \text{ and } (A_1 \wedge A_3) \vee (A_2 \wedge A_3) = 0, \\ 1 & \text{else,} \end{cases} \\
&= \begin{cases} A_1 \vee (A_2 \vee A_3) & \text{if } A_1 \wedge A_2 = 0,\ A_1 \wedge A_3 = 0, \text{ and } A_2 \wedge A_3 = 0, \\ 1 & \text{else,} \end{cases} \\
&= \begin{cases} A_1 \vee (A_2 \vee A_3) & \text{if } A_2 \wedge A_3 = 0 \text{ and } A_1 \wedge (A_2 \vee A_3) = 0, \\ 1 & \text{else,} \end{cases} \\
&= \begin{cases} A_1 + (A_2 \vee A_3) & \text{if } A_2 \wedge A_3 = 0, \\ A_1 + 1 & \text{else,} \end{cases} \\
&= A_1 + (A_2 + A_3).
\end{aligned}$$

Hence $(B^*, +)$ forms a semigroup, and is commutative by the commutativity of $\wedge$ and $\vee$. Note that $A + 1 = A \vee 1 = 1$ for all $A \in B^*$, so that 1 is the zero of $(B^*, +)$. If $A \in B^*$, then $A + A = 1$ as $A \wedge A = A$, and so $(B^*, +)$ forms a nil semigroup of degree 2. Moreover, by the last part of Remark 2.15 for any $A \in B^*$ we have $A = A_1 \vee A_2$ for some $A_1, A_2 \in B^*$ such that $A_1 \wedge A_2 = 0$. Hence $A = A_1 + A_2$, and so $(B^*, +)$ forms a zs-semigroup. Let us denote this semigroup by $[B]$. It is clear that if θ is an automorphism of B, then $\theta|_{B^*}$ is an automorphism of B^*. Since B is $\aleph_0$-categorical, so is $[B]$. □

We can say a little more: it is easy to build an example of an $\aleph_0$-categorical commutative nil semigroup that is nil of degree 2 and nilpotent but *not* nilpotent of degree 2.

Example 2.17 Let A be a countably infinite set, and let $u, 0$ be distinct symbols not in A. Let $C = A \cup \{0, u\}$ and define a binary operation on C by letting the only non-zero products be $ab = u$ where $a, b \in A$ and $a \neq b$. It is easy to see that C is a commutative semigroup, nil of degree 2 and nilpotent of degree 3. That C is $\aleph_0$-categorical follows easily from Proposition 2.11.

We say that a semigroup S is *characteristically simple* if it has no characteristic ideals other than $\emptyset$ or itself. Similarly, we say that a semigroup S with 0 is *characteristically 0-simple* if it has no characteristic ideals, other than $\emptyset$, $\{0\}$ and itself.

Proposition 2.18 *Let S be an $\aleph_0$-categorical semigroup. Then S is the union of a finite chain of characteristic subsemigroups*

$$S = S_0 \supset S_1 \supset \ldots \supset S_n$$

such that for $0 \leq i \leq n-1$, S_{i-1} is a characteristic ideal of S_i and the Rees quotients S_i/S_{i+1} are $\aleph_0$-categorical and characteristically 0-simple, and S_n is characteristically simple.

Proof For an $\aleph_0$-categorical semigroup S, let $\tau(S)$ denote $|S/\sim_{S,1}|$. Let U be a characteristic subsemigroup of S. Notice that for any $n \in \mathbb{N}$ and $\overline{u}, \overline{v} \in U^n$, if $\overline{u} \sim_{S,n} \overline{v}$, then $\overline{u} \sim_{U,n} \overline{v}$, since $U\phi = U$ for any $\phi \in \operatorname{Aut}(S)$. Hence if $U \neq S$ is a characteristic subsemigroup of S then U is $\aleph_0$-categorical by Corollary 2.14 (2) and $\tau(U) < \tau(S)$.

We proceed by induction on $\tau(S)$. If $\tau(S) = 1$, then certainly there are no proper characteristic ideals of S, so that the result is true with $n = 0$.

Suppose now that for any $\aleph_0$-categorical semigroup T with $\tau(T) < \tau(S)$ the result holds. Let T be a maximal proper characteristic ideal of S. If $T = \emptyset$ then we are done. Suppose therefore that $T \neq \emptyset$; the proof that S/T is $\aleph_0$-categorical follows as in (5) of Corollary 2.14 (see also (1) of Corollary 3.11).

If U is a proper characteristic ideal of S/T, then either $U = \{0\}$, or $U \backslash \{0\} \cup T$ is an ideal of S. Since $U \backslash \{0\}$ is a union of $\sim_{S/T,1}$-classes and hence of $\sim_{S,1}$-classes, we have that $(U\backslash\{0\}) \cup T$ is a characteristic ideal of S strictly containing T, a contradiction. Thus S/T is characteristically 0-simple.

From the first part of the proof we have $\tau(T) < \tau(S)$, so that, applying the result for T, we deduce the required sequence of ideals for S. □

The following example is clear.

Example 2.19 Let S be a semigroup. Then $E(S)$ and $\operatorname{Reg}(S)$ form characteristic subsets of S, where $\operatorname{Reg}(S)$ is the set of regular elements of S. The following subsemigroups of S (where they exist) are characteristic:

$$\langle E(S) \rangle, \{1\}, \{0\}, \langle \operatorname{Reg}(S) \rangle.$$

If S is commutative, then $E(S) = \langle E(S)\rangle$ forms a band and $\mathrm{Reg}(S) = \langle \mathrm{Reg}(S)\rangle$ forms a semilattice of abelian groups, that is, a commutative Clifford semigroup (see [19, Chapter IV]).

We address the $\aleph_0$-categoricity of Clifford semigroups in the sequel [29]. Proposition 2.20 below gives a taster of the results for Clifford semigroups, in the special case where the connecting homomorphisms are trivial.

Proposition 2.20 *Let $S = \bigcup_{i\in Y} S_i$ be a finite chain of semigroups such that for any $i > j$, $s_i \in S_i$, $s_j \in S_j$ we have $s_i s_j = s_j = s_j s_i$. If each S_i is $\aleph_0$-categorical then S is $\aleph_0$-categorical. Moreover, if each S_i is characteristic (for example, if it is a non-trivial group) then the converse holds.*

Proof Let us refer to the S_i $(i \in Y)$ as the components of S.

Suppose each S_i is $\aleph_0$-categorical. Let $n \in \mathbb{N}$ and notice that in any infinite list of elements of S^n we can pick a sublist $(a_1^i, \ldots, a_n^i)$ such that for any $1 \leq \ell \leq n$ the elements $a_\ell^1, a_\ell^2, \ldots$ all lie in the same component of S. Without loss of generality, suppose that $a_1^k, \ldots, a_{j_1}^k \in S_{i_1}, a_{j_1+1}^k, \ldots, a_{j_2}^k \in S_{i_2}, \ldots, a_{j_{u-1}+1}^k, \ldots, a_n^k \in S_{i_u}$. Since each S_k is $\aleph_0$-categorical, we may find an $i < j$ and $\phi_\ell \in \mathrm{Aut}(S_{i_\ell})$, $1 \leq \ell \leq u$, such that $\phi' = \bigcup_{1\leq \ell\leq u} \phi_\ell$ takes $(a_1^i, \ldots, a_n^i)$ to $(a_1^j, \ldots, a_n^j)$. For any $t \in Y\backslash\{i_1, \ldots, i_u\}$, let $\phi_t = I_{S_t}$. It is easy to see that $\phi = \bigcup_{i\in Y} \phi_i$ lies in $\mathrm{Aut}(S)$ and clearly takes $(a_1^i, \ldots, a_n^i)$ to $(a_1^j, \ldots, a_n^j)$.

The converse is clear. □

Finally in this section we make a comment concerning chains of [one-sided] ideals of S. Recall that in a partially ordered set L, an element u *covers* an element v, written $v \prec u$, if $v < u$ and for all w with $v \leq w \leq u$ we have $v = w$ or $w = u$. If we have a chain of elements in L such that each element covers its predecessor, then we call this a *covering chain*.

Proposition 2.21 *Let S be an $\aleph_0$-categorical semigroup in which the principal right [left, two-sided] ideals form a chain. Then there are no infinite ascending or descending covering chains of principal right [left, two-sided] ideals.*

Proof We argue for ascending chains of principal right ideals, the other cases being similar.

Suppose that $a_1, a_2, \ldots \in S$ and

$$a_1 S^1 \prec a_2 S^1 \prec \ldots .$$

Then (a_1, a_i) for $i \in \mathbb{N}$ lie in different 2-automorphism types, a contradiction. □

As the case of a dense linear order shows, we cannot expect to have full ascending or descending chain conditions on ideals in $\aleph_0$-categorical semigroups.

3 Inherited categoricity

We remarked in Corollary 2.14 that $\aleph_0$-categoricity is inherited by characteristic subsemigroups. We note that $\aleph_0$-categoricity is not inherited by every subsemigroup, and an example for groups is given by Olin in [27]. However, the condition that a subsemigroup be characteristic to inherit $\aleph_0$-categoricity is too restrictive; since many key subsemigroups, such as maximal subgroups and principal ideals, are not necessarily characteristic. The components in a finite chain of groups as in Proposition 2.20 are, but this relies on the chain being finite. We thus study a weaker condition for a subsemigroup that still guarantees the preservation of $\aleph_0$-categoricity.

Definition 3.1 Let S be a semigroup and, for some fixed $t \in \mathbb{N}$, let $\{\overline{X}_i : i \in I\}$ be a collection of t-tuples of S. Let $\{A_i : i \in I\}$ be a collection of subsets [subsemigroups, ideals] of S with the property that for any automorphism ϕ of S such that there exists $i, j \in I$ with $\overline{X}_i\phi = \overline{X}_j$, then $\phi|_{A_i}$ is a bijection from A_i onto A_j. Then we call $\mathcal{A} = \{(A_i, \overline{X}_i) : i \in I\}$ a *system of t-pivoted pairwise relatively characteristic (t-pivoted p.r.c.) subsets [subsemigroups, ideals]* of S. The t-tuple $\overline{X}_i$ is called the *pivot* of A_i $(i \in I)$. If $|I| = 1$ then, letting $A_1 = A$ and $\overline{X}_1 = \overline{X}$, we write $\{(A, \overline{X})\}$ simply as $(A, \overline{X})$, and call A an *$\overline{X}$-pivoted relatively characteristic ($\overline{X}$-pivoted r.c.) subset [subsemigroup, ideal]* of S.

Clearly if $\{(A_i, \overline{X}_i) : i \in I\}$ forms a system of t-pivoted p.r.c. subsets of S and J is a subset of I then $\{(A_j, \overline{X}_j) : j \in J\}$ is also a system of t-pivoted p.r.c. subsets of S. In particular, each A_i is an $\overline{X}_i$-pivoted r.c. subset of S. Moreover, if A is an $\overline{X}$-pivoted r.c. subset of S then A is a union of orbits of the set of automorphisms of S which fix $\overline{X}$, since if $a \in A$ and $\phi \in \mathrm{Aut}(S)$ fixes $\overline{X}$ then $A\phi = A$, so that $a\phi \in A$.

Definition 3.1 has strong links with the model theoretic concept of *definability*, and we refer the reader to the introduction of [10] for a background into these links. In fact much of the work in this section could be given in terms of definable sets, but in keeping with our algebraic viewpoint it is more natural to use Definition 3.1.

Lemma 3.2 *Let S be a semigroup and, for some fixed $t \in \mathbb{N}$, let $\{\overline{X}_i : i \in I\}$ be a collection of t-tuples of S. Then for any collection $\{A_i : i \in I\}$ of subsets of S, the following are equivalent:*

(1) $\{(A_i, \overline{X}_i) : i \in I\}$ is a system of t-pivoted p.r.c. subsets [subsemigroups, ideals] of S;
(2) if $\phi \in \mathrm{Aut}(S)$ is such that there exists $i, j \in I$ with $\overline{X}_i\phi = \overline{X}_j$, then $A_i\phi \subseteq A_j$.

Proof This follows immediately from applying the definitions, and the fact that if $\phi \in \mathrm{Aut}(S)$ and $\overline{X}_i\phi = \overline{X}_j$, then $\phi^{-1} \in \mathrm{Aut}(S)$ and $\overline{X}_j\phi^{-1} = \overline{X}_i$. □

Consequently, if $\{(A_i, \overline{X}_i) : i \in I\}$ is a system of t-pivoted p.r.c. subsets of a semigroup S then $\{(\langle A_i \rangle, \overline{X}_i) : i \in I\}$ forms a system of t-pivoted p.r.c. subsemigroups of S. For if $\phi \in \mathrm{Aut}(S)$ is such that $\overline{X}_i\phi = \overline{X}_j$ for some $i, j \in I$ then $A_i\phi = A_j$, and so $\langle A_i \rangle\phi \subseteq \langle A_j \rangle$. The result follows by Lemma 3.2.

Notation Given a pair of tuples $\overline{a} = (a_1, \ldots, a_n)$ and $\overline{b} = (b_1, \ldots, b_m)$, we denote $(\overline{a}, \overline{b})$ as the $(n+m)$-tuple given by

$$(a_1, \ldots, a_n, b_1, \ldots, b_m).$$

Proposition 3.3 *Let S be an $\aleph_0$-categorical semigroup and $\{(A_i, \overline{X}_i) : i \in I\}$ be a system of t-pivoted p.r.c. subsets of S. Then $\{|A_i| : i \in I\}$ is finite. If, further, each A_i forms a subsemigroup of S, then $\{A_i : i \in I\}$ is finite, up to isomorphism, with each A_i being $\aleph_0$-categorical.*

Proof Suppose for some $i \neq j$ we have $\overline{X}_i \sim_{S,t} \overline{X}_j$ via $\phi \in \mathrm{Aut}(S)$, say. Then $A_i\phi = A_j$ and it follows that both $|\{|A_i| : i \in I\}|$ and number of non-isomorphic elements of $\{A_i : i \in I\}$ is bound by the number of t-automorphism types of S, which is finite by the $\aleph_0$-categoricity of S.

Suppose A_i forms a subsemigroup of S. Let $\overline{X}_i = (x_{i1}, \ldots, x_{it})$, and suppose $\overline{a} = (a_1, \ldots, a_n)$ and $\overline{b} = (b_1, \ldots, b_n)$ are a pair of n-tuples of A_i such that $(\overline{a}, \overline{X}_i) \sim_{S,n+t} (\overline{b}, \overline{X}_i)$ via $\phi \in \mathrm{Aut}(S)$, say. Then $\overline{X}_i\phi = \overline{X}_i$ and so $\phi|_{A_i}$ is an automorphism of A_i as $(A_i, \overline{X}_i)$ is a t-pivoted r.c. subsemigroup. Moreover, $\overline{a}\phi|_{A_i} = \overline{a}\phi = \overline{b}$ and so $\overline{a} \sim_{A_i,n} \overline{b}$. We have thus shown that

$$|A_i^n / \sim_{A_i,n}| \leq |S^{n+t} / \sim_{S,n+t}| < \aleph_0$$

for each $n \in \mathbb{N}$, since S is $\aleph_0$-categorical. Hence A_i is $\aleph_0$-categorical by the RNT. $\square$

Corollary 3.4 *Let S be an $\aleph_0$-categorical semigroup. Then there are only finitely many principal [left, right] ideals, up to isomorphism, and these are all $\aleph_0$-categorical.*

Proof We show that $\{(S^1aS^1, a) : a \in S\}$ forms a system of 1-pivoted p.r.c. subsets of S. To see this, let $\phi \in$ Aut(S) be such that $a\phi = b$, and let $x \in S^1aS^1$. Then there exists $u, v \in S^1$ with $x = uav$, and so by interpreting 1ϕ as 1 we have

$$x\phi = (u\phi)(a\phi)(v\phi) = (u\phi)b(v\phi) \in S^1bS^1,$$

and the result follows by Lemma 3.2. A similar result holds for principal left/right ideals. $\square$

Motivated by Green's relations, we now give a method for constructing systems of t-pivoted p.r.c. subsets of a semigroup via certain equivalence relations. Let $\phi : S \to T$ be an isomorphism between semigroups S and T, and τ_S and τ_T be equivalence relations on S and T, respectively. We call τ_S and τ_T *preserved by ϕ* if $a\, \tau_S\, b$ if and only if $a\phi\, \tau_T\, b\phi$ for each $a, b \in S$. This is clearly equivalent to

$$(x\tau_S)\phi = (x\phi)\tau_T \quad (\forall x \in S),$$

where ϕ is applied pointwise, and yet again to $\phi' : S/\tau_S \to T/\tau_T$ given by

$$[x]_{\tau_S}\phi' = [x\phi]_{\tau_T} \quad (\forall x \in S)$$

being a well-defined bijection. If $S = T$, so that ϕ is an automorphism of S, then we say that τ_S is *preserved by ϕ*.

Note that if τ is an equivalence relation on a semigroup S then

$$\mathrm{Aut}(S)[\tau] := \{\phi \in \mathrm{Aut}(S) : \tau \text{ is preserved by } \phi\}$$

is a subgroup of $\mathrm{Aut}(S)$.

If $\operatorname{Aut}(S) = \operatorname{Aut}(S)[\tau]$ then we call τ *preserved by automorphisms* (of S).

Example 3.5 (1) For any semigroup S, if U is a characteristic subset, then $\langle U \times U \rangle$ is preserved by automorphisms.
(2) If S is an inverse semigroup, then the least group congruence σ on S given by

$$a \,\sigma\, b \Leftrightarrow (\exists e \in E(S)) \quad ea = eb,$$

is preserved by automorphisms.
(3) If ρ is a relation preserved by automorphisms, then so too are the congruences $\rho^\sharp = \langle \rho \rangle$ and $\rho^\flat$, where $\rho^\flat$ is the largest congruence contained in ρ.
(4) If S is an inverse semigroup then μ, the maximum idempotent-separating congruence on S, is preserved by automorphisms.

Proof Statements (1) and (3) are clear. We remark that if σ is given by the formula in (2), then $\sigma = \langle E(S) \times E(S) \rangle$, and for (4) we require the fact that for any inverse semigroup S, we have $\mu = \mathcal{H}^\flat$ [19, Proposition 5.3.7]. $\square$

The following lemma is then immediate from Proposition 3.3.

Lemma 3.6 *Let S be a semigroup and τ be an equivalence relation on S, preserved by automorphisms of S. Then $\{(x\tau, x) : x \in S\}$ forms a system of 1-pivoted p.r.c. subsets of S. Hence if S is $\aleph_0$-categorical then there are only finitely many cardinalities of τ-classes and only finitely many subsemigroup τ-classes, up to isomorphism.*

Corollary 3.7 *Let S be an $\aleph_0$-categorical semigroup. Then for any of Green's relations $\mathcal{K}$, we have*

$$|\{|K_a| : a \in S\}| < \aleph_0.$$

Moreover, if U is a transversal of the set of $\mathcal{K}$-classes that are subsemigroups, there are only finitely many K_u-classes ($u \in U$), up to isomorphism, and each K_u is $\aleph_0$-categorical.

In particular, there are only finitely many maximal subgroups, up to isomorphism, and each of these is $\aleph_0$-categorical.

Proof Each Green's relation is preserved by automorphisms. Consequently, for any semigroup S and any $K \in \{\mathcal{R}, \mathcal{L}, \mathcal{H}, \mathcal{D}, \mathcal{J}\}$, we have $\{(K_a, a) : a \in S\}$ as a system of 1-pivoted p.r.c. subsets of S. The result then follows from Lemma 3.6. $\square$

A similar statement to the above also holds for Green's *-relations, and Green's $\sim$-relations [14]. It is worth exercising some caution here. In the corollary above the maximal subgroups are $\aleph_0$-categorical semigroups, while earlier investigations into the $\aleph_0$-categoricity of groups considered them as a set with a single binary operation, a single unary operation (inverse), and a single constant (the identity). However, since a semigroup automorphism of a group is necessarily a group automorphism, it follows from the RNT that our two concepts of $\aleph_0$-categoricity of a group coincide, and we can write $\aleph_0$-*categorical group* without ambiguity.

Much like the situation with characteristic subsets, for results relating to inherited $\aleph_0$-categoricity of quotients we require only that congruences are preserved by all automorphisms fixing a finite number of elements. This leads us to the following definition.

Definition 3.8 Let τ be an equivalence relation on a semigroup S and $\overline{X}$ be an n-tuple of S for some $n \in \mathbb{N}^0$. We say that τ is *$\overline{X}$-relatively automorphism preserved* (*$\overline{X}$-r.a.p.*) *with pivot* $\overline{X}$, if whenever $\phi \in \text{Aut}(S)$ is such that $\overline{X}\phi = \overline{X}$, then ϕ preserves τ.

If $\overline{X}$ is the empty tuple then τ being $\overline{X}$-r.a.p. is equivalent to τ being preserved by automorphisms. Moreover, as with $\overline{X}$-pivoted r.c. subsets, there exists connections between definable sets of ordered pairs of a semigroup and $\overline{X}$-r.a.p. equivalence relations.

Lemma 3.9 *Let S be a semigroup, let $\overline{X} \in S^t$ for some $t \in \mathbb{N}^0$, and τ an $\overline{X}$-r.a.p. equivalence relation on S. For each $a \in S$, let $\overline{X}_a$ be the $(t+1)$-tuple given by $(\overline{X}, a)$. Then $\{(a\tau, \overline{X}_a) : a \in S\}$ forms a system of $(t+1)$-pivoted p.r.c. subsets of S.*

Proof Let ϕ be an automorphism of S such that $\overline{X}_a\phi = \overline{X}_b$ for some $a, b \in S$. Then $\overline{X}\phi = X$ so that τ is preserved by ϕ, and $a\phi = b$. Hence

$$(a\tau)\phi = b\tau. \qquad \square$$

Note that by letting $\overline{X}$ be the empty tuple in the lemma above we recover the first statement in Lemma 3.6.

Our next aim is to use the results above to assess when the $\aleph_0$-categoricity of a semigroup passes to its quotients.

Proposition 3.10 *Let S be an $\aleph_0$-categorical semigroup, let $\overline{X} \in S^t$ for some $t \in \mathbb{N}^0$, and ρ an $\overline{X}$-r.a.p. congruence on S. Then S/ρ is $\aleph_0$-categorical.*

Proof Suppose $\overline{X} \in S^t$ and let $\overline{a} = (a_1\rho, \ldots, a_n\rho)$ and $\overline{b} = (b_1\rho, \ldots, b_n\rho)$ be a pair of n-tuples of S/ρ such that $(a_1, \ldots, a_n, \overline{X}) \sim_{S,n+t} (b_1, \ldots, b_n, \overline{X})$ via $\phi \in \text{Aut}(S)$, say. Then $\overline{X}\phi = \overline{X}$, so that ρ is preserved by the automorphism ϕ, and there is thus an automorphism ψ of S/ρ given by

$$[a]_\rho\psi = [a\phi]_\rho \quad (a\rho \in S/\rho).$$

Since $[a_k]_\rho\psi = [a_k\phi]_\rho = [b_k]_\rho$ for each $1 \leq k \leq n$, we have $\overline{a} \sim_{S/\rho,n} \overline{b}$, and so

$$|(S/\rho)^n/ \sim_{S/\rho,n} | \leq |S^{n+t}/ \sim_{S,n+t} | < \aleph_0,$$

as S is $\aleph_0$-categorical. Hence S/ρ is $\aleph_0$-categorical. $\square$

If we drop the condition on Proposition 3.10 that the congruence is relatively automorphism preserving then the statement is no longer true. An example of an $\aleph_0$-categorical group with a non-$\aleph_0$-categorical quotient group is given by Rosenstein [35].

Corollary 3.11 *Let S be an $\aleph_0$-categorical semigroup.*

(1) If ρ is a congruence preserved by automorphisms, then S/ρ is $\aleph_0$-categorical.
(2) If S is inverse, then S/σ is an $\aleph_0$-categorical group.
(3) The semigroup $S/\mathcal{H}^\flat$ is $\aleph_0$-categorical, so that if S is inverse, then S/μ is $\aleph_0$-categorical.
(4) If ρ is a finitely generated congruence, then S/ρ is $\aleph_0$-categorical.
(5) If I is an $\overline{X}$-pivoted r.c. ideal of S for some finite tuple $\overline{X}$ of elements of S, then S/I is $\aleph_0$-categorical.

Proof (1)–(3) follow from Example 3.5 and Proposition 3.10. For (4) we let $\rho = \langle (u_1, v_1), \ldots, (u_r, v_r) \rangle$ be a finitely generated congruence on S and let $\overline{X} = (u_1, v_1, \ldots, u_n, v_n)$. It is easy to see from the explicit description of ρ (see [19, Proposition 1.5.9]) that ρ is an $\overline{X}$-pivoted r.a.p. congruence with pivot $\overline{X}$. The result is then immediate from Proposition 3.10.

For (5), suppose that I is an $\overline{X}$-p.r.c. ideal of S. Let ϕ be an automorphism of S which fixes $\overline{X}$, so that $I\phi = I$ since I is an $\overline{X}$-pivoted r.c. ideal. Then, for any $a, b \in S$, we have

$$a \, \rho_I \, b \Leftrightarrow [a = b \text{ or } a, b \in I] \Leftrightarrow [a\phi = b\phi \text{ or } a\phi, b\phi \in I] \Leftrightarrow a\phi \, \rho_I \, b\phi,$$

thus showing that ρ_I is an $\overline{X}$-r.a.p. congruence, and once more we call upon Proposition 3.10. □

If S is a semigroup, then it is clear the intersection of all non-empty ideals is an ideal. If this is non-empty, it is denoted by $K(S)$. Thus $K(S)$, if it exists, is the unique minimum non-empty ideal of S.

Theorem 3.12 *The principal factors of an $\aleph_0$-categorical semigroup S are $\aleph_0$-categorical, and either completely 0-simple, completely simple or null. Moreover, S has only finitely many principal factors, up to isomorphism.*

Proof For each $a \in S$ let $J(a) = S^1 a S^1$ and $I(a) = J(a) \backslash J_a$. As in Corollary 3.4, $\{(J(a), a) : a \in S\}$ is a system of 1-pivoted p.r.c. subsemigroups of S and, since S is $\aleph_0$-categorical, the ideals $J(a)$ are $\aleph_0$-categorical. Let ϕ be an automorphism of S such that $a\phi = b$, and so $J(a)\phi = J(b)$. Moreover, as $\mathcal{J}$ is preserved by automorphisms we have $J_a \phi = J_b$, and so

$$I(a)\phi = (J(a) \backslash J_a)\phi = J(b) \backslash J_b = I(b).$$

Consequently, $\{(I(a), a) : a \in S\}$ is a system of 1-pivoted p.r.c. subsemigroups of S and, in particular, $I(a)$ is an a-pivoted r.c. ideal of $J(a)$ for each $a \in S$. Hence $J(a)/I(a)$ is $\aleph_0$-categorical by Corollary 3.11 (5). If $K(S)$ exists, then it is a $\mathcal{J}$-class of S, and is thus $\aleph_0$-categorical. Hence each principal factor of S is $\aleph_0$-categorical.

Moreover, as $\phi|_{J(a)}$ is an isomorphism from $J(a)$ to $J(b)$ with $I(a)\phi|_{J(a)} = I(b)$, it follows that the isomorphism $\phi|_{J(a)}$ preserves $\rho_{I(a)}$ and $\rho_{I(b)}$, and so ϕ induces an isomorphism from $J(a)/I(a)$ to $J(b)/I(b)$. Hence the set $\{J(a)/I(a) : a \in S\}$ of non-kernel principal factors of S has at most $|S/\sim_{S,1}|$ elements, up to isomorphism. Since $K(S)$ is unique, S has only finitely many principal factors, up to isomorphism.

By [5, Lemma 2.39], the principal factors of S are either 0-simple, simple or null. A periodic [0-]simple semigroup is completely [0-]simple (the result for 0-simple semigroups is given in [5, Corollary 2.56], from which the simple case follows). Hence as an $\aleph_0$-categorical semigroup is periodic by Corollary 2.3, each principal factor is either completely 0-simple, completely simple or null. $\square$

Recall that every null semigroup is $\aleph_0$-categorical by Example 2.7. To understand the $\aleph_0$-categoricity of an arbitrary semigroup it is therefore essential to examine the completely simple and completely 0-simple cases. The $\aleph_0$-categoricity of inverse completely 0-simple semigroups will be considered in Sect. 4, while the $\aleph_0$-categoricity of an arbitrary completely [0-]simple semigroup will be the main topic of a subsequent paper [29].

As we have seen in Proposition 2.11 and Corollary 3.11, the RNT is adept at dealing with a range of finiteness conditions. We end this section by studying a final finiteness condition: equivalence relations on a semigroup with finite equivalence classes.

Let S be a semigroup and τ an equivalence relation on S. For each $n \in \mathbb{N}$, define an equivalence relation $\#_{S,\tau,n}$ on S^n by $(a_1, \ldots, a_n)\, \#_{S,\tau,n}\, (b_1, \ldots, b_n)$ if and only if there exists an automorphism ϕ of S such that $(a_k\tau)\phi = b_k\tau$ for each $1 \leq k \leq n$.

Proposition 3.13 *Let S be a semigroup and τ an equivalence relation on S with each τ-class being finite. Then $|S^n/\#_{S,\tau,n}|$ is finite for each $n \in \mathbb{N}$ if and only if S is $\aleph_0$-categorical and $A = \{|m\tau| : m \in S\}$ is finite.*

Proof Suppose that $|S^n/\#_{S,\tau,n}|$ is finite for each $n \in \mathbb{N}$. Let $Z = \{\overline{a}_i : i \in \mathbb{N}\}$ be an infinite set of n-tuples of S, where $\overline{a}_i = (a_{i1}, \ldots, a_{in})$. Since $|S^n/\#_{S,\tau,n}|$ is finite, there exists an infinite subset $\{\overline{a}_i : i \in I\}$ of Z such that $\overline{a}_i\, \#_{S,\tau,n}\, \overline{a}_j$ for each $i, j \in I$. Fix some $p \in I$. Then for each $i \in I$ there exists an automorphism ϕ_i of S with $(a_{ik}\tau)\phi_i = a_{pk}\tau$ for each $1 \leq k \leq n$. Hence $a_{ik}\phi_i \in a_{pk}\tau$ for each $1 \leq k \leq n$, so that

$$\overline{a}_i\phi_i \in \{(z_1, \ldots, z_n) : z_k \in a_{pk}\tau\}.$$

Notice that the set $\{(z_1, \ldots, z_n) : z_k \in a_{pk}\tau\}$ is finite since each τ-class is finite. Consequently, there exists distinct $i, j \in I$ such that $\overline{a}_i\phi_i = \overline{a}_j\phi_j$, so that $\overline{a}_i\phi_i\phi_j^{-1} = \overline{a}_j$. Hence $\overline{a}_i$ and $\overline{a}_j$ are automorphically equivalent. It follows that S contains no infinite set of distinct n-automorphism types, and is thus $\aleph_0$-categorical by the RNT. Furthermore, by our usual argument we have that $|A|$ is bound by $|S/\#_{S,\tau,1}|$.

Conversely, suppose S is $\aleph_0$-categorical and A is finite. Let $\overline{m} = (m_1, \ldots, m_n)$ and $\overline{m}' = (m_1', \ldots, m_n')$ be a pair of n-tuples of S, under the condition that $|m_k\tau| = |m_k'\tau|$ for each k. Since each entry of an n-tuple of S^n has $|A|$ potential cardinalities for its τ-class, it follows that this condition has $|A|^n$ choices. For each $1 \leq k \leq n$, let $m_k\tau = \{a_{k1}, \ldots, a_{ks_k}\}$ and $m_k'\tau = \{b_{k1}, \ldots, b_{ks_k}\}$, and let $T(n) = s_1 + s_2 + \cdots + s_n$. Suppose further that

$$(a_{11}, \ldots, a_{1s_1}, a_{21}, \ldots, a_{2s_2}, \ldots, a_{ns_n}) \sim_{S,T(n)} (b_{11}, \ldots, b_{1s_1}, b_{21}, \ldots, b_{2s_2}, \ldots, b_{ns_n}),$$

via $\phi \in \mathrm{Aut}(S)$, say. Note that this condition also has finitely many choices as $|S^{T(n)}/\sim_{S,T(n)}|$ is finite for each $n \in \mathbb{N}$ by the RNT. Moreover, $(m_k\tau)\phi = m'_k\tau$ for each k, since $a_{kr}\phi = b_{kr}$ for each $1 \leq r \leq s_k$. Hence $\overline{m} \#_{S,\tau,n} \overline{m}'$, and so $|S^n/\#_{S,\tau,n}|$ is finite by Lemma 2.8. □

Corollary 3.14 *Let S be a regular semigroup with each maximal subgroup being finite. Then S is $\aleph_0$-categorical if and only if $|E(S)^n/\sim_{S,n}|$ is finite for each $n \in \mathbb{N}$.*

Proof If S is $\aleph_0$-categorical, then

$$|E(S)^n/\sim_{S,n}| \leq |S^n/\sim_{S,n}| < \aleph_0$$

for each $n \in \mathbb{N}$ by the RNT.

Conversely, suppose $|E(S)^n/\sim_{S,n}|$ is finite for each $n \in \mathbb{N}$ and consider a pair of n-tuples of S given by $\overline{a} = (a_1, \ldots, a_n)$ and $\overline{b} = (b_1, \ldots, b_n)$. Since S is regular, there exists idempotents $e_i, f_i, \bar{e}_i, \bar{f}_i$ of S with $e_i \,\mathcal{R}\, a_i \,\mathcal{L}\, f_i$ and $\bar{e}_i \,\mathcal{R}\, b_i \,\mathcal{L}\, \bar{f}_i$ for each $1 \leq i \leq n$. Suppose further that

$$(e_1, f_1, e_2, f_2, \ldots, e_n, f_n) \sim_{S,2n} (\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \ldots, \bar{e}_n, \bar{f}_n),$$

via $\phi \in \mathrm{Aut}(S)$, say. Then as $\mathcal{R}$ and $\mathcal{L}$ are automorphism preserving we have that $R_{e_i}\phi = R_{\bar{e}_i}$ and $L_{f_i}\phi = L_{\bar{f}_i}$ for each i, so that

$$H_{a_i}\phi = (R_{a_i} \cap L_{a_i})\phi = (R_{e_i} \cap L_{f_i})\phi = R_{e_i}\phi \cap L_{f_i}\phi = R_{\bar{e}_i} \cap L_{\bar{f}_i} = H_{b_i}.$$

Hence $\overline{a} \#_{S,\mathcal{H},n} \overline{b}$, and we have thus shown that

$$|S^n/\#_{S,\mathcal{H},n}| \leq |E(S)^{2n}/\sim_{S,2n}| < \aleph_0.$$

Since each maximal subgroup of S is finite, every $\mathcal{H}$-class of S is finite by the regularity of S and [19, Lemma 2.2.3]. Hence S is $\aleph_0$-categorical by Proposition 3.13. □

4 Building $\aleph_0$-categorical semigroups: Brandt semigroups, direct sums and 0-direct unions

In this section we consider the $\aleph_0$-categoricity of a number of well known constructions, the first being motivated by Theorem 3.12.

The *Brandt semigroup over a group G with index set I*, denoted $\mathcal{B}^0[G; I]$, is the set $(I \times G \times I) \cup \{0\}$ with multiplication $(i, g, j)0 = 0(i, g, j) = 00 = 0$ and

$$(i, g, j)(k, h, l) = \begin{cases} (i, gh, l) & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

Every Brandt semigroup is an inverse completely 0-simple semigroup and, conversely, an inverse completely 0-simple semigroup is isomorphic to some Brandt semigroup [19, Theorem 5.1.8]. Our early interest in Brandt semigroups from an $\aleph_0$-categorical perspective is due to the simplicity with which their automorphisms may be determined. The automorphism theorem for Brandt semigroups below is a direct consequence of [19, Theorem 3.4.1]:

Proposition 4.1 *Let $S = \mathcal{B}^0[G; I]$ be a Brandt semigroup. Let θ be an automorphism of G, and π a bijection of I. Then the map $\psi : S \to S$ given by $0\psi = 0$ and $(i, g, j)\psi = (i\pi, g\theta, j\pi)$ for each $(i, g, j) \in S\backslash\{0\}$ is an automorphism, denoted $\psi = (\theta; \pi)$. Conversely, every automorphism of $\mathcal{B}^0[G; I]$ may be constructed in this manner.*

Theorem 4.2 *A Brandt semigroup $S = \mathcal{B}^0[G; I]$ is $\aleph_0$-categorical if and only if G is $\aleph_0$-categorical.*

Proof ($\Rightarrow$) Since G is isomorphic to each non-zero maximal subgroup $B_i = \{(i, g, i) : g \in G\}$ of S, the result follows from Corollary 3.7.

($\Leftarrow$) By the RNT and Proposition 2.11, to prove the $\aleph_0$-categoricity of S it suffices to show that the number of n-automorphism types of $S^* = S\backslash\{0\}$ is finite for each $n \in \mathbb{N}$. Let $\overline{a} = ((i_1, g_1, j_1), \ldots, (i_n, g_n, j_n))$ and $\overline{b} = ((k_1, h_1, \ell_1), \ldots, (k_n, h_n, \ell_n))$ be a pair of n-tuples of S^* such that

(1) $(i_1, \ldots, i_n, j_1, \ldots, j_n) \natural_{I,2n} (k_1, \ldots, k_n, \ell_1, \ldots, \ell_n)$,
(2) $(g_1, \ldots, g_n) \sim_{G,n} (h_1, \ldots, h_n)$, via $\theta \in \mathrm{Aut}(G)$, say.

By condition (1) there exists a bijection π from $\{i_1, \ldots, i_n, j_1, \ldots, j_n\}$ to $\{k_1, \ldots, k_n, \ell_1, \ldots, \ell_n\}$ given by $i_r\pi = k_r$ and $j_r\pi = \ell_r$ $(1 \leq r \leq n)$. Moreover, condition (1) has B_{2n} choices, which is finite. By the $\aleph_0$-categoricity of G, condition (2) also has finitely many choices. Take a bijection $\bar{\pi}$ of I which extends π. Then $\psi = (\theta; \bar{\pi})$ is an automorphism of S by Proposition 4.1, and is such that

$$(i_r, g_r, j_r)\psi = (i_r\bar{\pi}, g_r\theta, j_r\bar{\pi}) = (i_r\pi, h_r, j_r\pi) = (k_r, h_r, \ell_r)$$

for each $1 \leq r \leq n$. Hence $\overline{a} \sim_{S,n} \overline{b}$, and so S is $\aleph_0$-categorical by Lemma 2.8. □

The classification of $\aleph_0$-categorical Brandt semigroups is an example of building $\aleph_0$-categorical semigroups from $\aleph_0$-categorical 'ingredients', in this case groups (and sets). The rest of the article is attributed to investigating constructions of this form for a number of rudimentary examples, the next being direct sums.

Let I be an indexing set and suppose that for each $i \in I$ we have a monoid M_i with identity 1_i. By the *direct sum* $S = \bigoplus_{i \in I} M_i$ we mean the submonoid

$$\{(m_i)_{i \in I} : m_i = 1_i \text{ for all but finitely many } i \in I\}$$

of the direct product $P = \Pi_{i \in I} M_i$. Rosenstein [34] showed that any group that is a direct sum of copies of finitely many finite groups is $\aleph_0$-categorical if and only if every group which occurs infinitely often in the sum is abelian. For our purposes we extend his result slightly as follows. Recall that the *centre* $Z(G)$ of a group G is the subgroup of G consisting of all those $a \in G$ such that $ag = ga$ for all $g \in G$.

Lemma 4.3 (cf. [34, Theorem 3]) *Let $S = \bigoplus_{i\in\mathbb{N}} M_i$ where each M_i is a finite group. Then if S is $\aleph_0$-categorical, all but finitely many of the M_i's are abelian.*

Proof Suppose for a contradiction that

$$M_{i_1}, M_{i_2}, \ldots$$

are all non-abelian. Pick $a_j \in M_{i_j}$ with $a_j \notin Z(M_{i_j})$, so that $|\{b^{-1}a_j b : b \in M_{i_j}\}| = m_j > 1$. Let $\overline{s_j} \in S$ be defined by

$$(\overline{s_j})_{i_k} = a_k \text{ for } 1 \leq k \leq j,\ (\overline{s_j})_i = 1_i \text{ else.}$$

Then $|\{\overline{b}^{-1}\overline{s_j}\overline{b} : \overline{b} \in S\}| = m_1 m_2 \ldots m_j$ so that for $i \neq j$ we cannot have $(\overline{s_i}) \sim_{S,1} (\overline{s_j})$, contradicting the $\aleph_0$-categoricity of S. □

Theorem 4.4 *Let $S = \bigoplus_{i\in\mathbb{N}} M_i$ be a direct sum of finite monoids M_i. Then S is $\aleph_0$-categorical if and only if S is a direct product of a finite monoid and an abelian group of bounded order.*

Proof Suppose first that S is $\aleph_0$-categorical. We first show that all but finitely many of the monoids M_i are groups. Suppose we have an infinite sequence

$$M_{i_1}, M_{i_2}, \ldots$$

such that each M_{i_j} is not a group. By this hypothesis, for each $j \in \mathbb{N}$ we may choose a non-identity idempotent $e_j \in M_{i_j}$ such that e_j is maximal in $E(M_{i_j})\backslash\{1_{i_j}\}$ (under the natural partial order given by $e \leq f$ if and only if $ef = fe = e$).

Consider the sequence

$$\overline{s_1}, \overline{s_2}, \overline{s_3}, \ldots$$

where

$$(\overline{s_j})_{i_k} = e_k \text{ for } 1 \leq k \leq j,\ (\overline{s_j})_i = 1_i \text{ else.}$$

For each $j \in \mathbb{N}$ there are exactly 2^j idempotents of S equal to or greater than $\overline{s_j}$ in the natural partial order, so that the elements $\overline{s_j}$ lie in distinct $\sim_{S,1}$-classes, a contradiction. Thus $S = M \times G$ where M is a finite monoid and G is a direct sum of finite groups.

The group of units of S is $\aleph_0$-categorical by Corollary 3.7, and is a direct sum of G and the group of units H of M. By Lemma 4.3 all but finitely many of the constituents of the direct sum forming G are abelian, so that $G = K \times W$ where K is finite and W is an abelian group of bounded order, hence $\aleph_0$-categorical by [34, Theorem 2]. Thus $S = M \times K \times W$ where $M \times K$ is finite and W is an abelian group of bounded order.

The converse is clear as $\aleph_0$-categoricity is preserved by finite direct products by [16]. □

To translate to the semigroup case requires some care, as here the direct sum does not embed into the direct product. Let M_i be a semigroup for each $i \in I$. By the *direct sum* of the semigroups M_i $(i \in I)$ we mean the semigroup

$$S = \left\langle \bigcup_{i \in I} \underline{M_i} : \underline{m_i}\,\underline{m_i'} = \underline{m_i m_i'},\ \underline{m_i}\,\underline{m_j} = \underline{m_j}\,\underline{m_i}\ \forall i \neq j, m_i, m_i' \in M_i, m_j \in M_j \right\rangle,$$

where $\underline{M_i} = \{\underline{m_i} : m_i \in M_i\}$ for all $i \in I$.

Proposition 4.5 *Let S be the direct sum of the finite semigroups M_i $(i \in I)$. Then S is $\aleph_0$-categorical if and only if I (and hence S) is finite.*

Proof Suppose that S is $\aleph_0$-categorical. For each M_i we choose a maximal idempotent e_i. If I is infinite, then without loss of generality we may take $I = \mathbb{N}$. Let $\underline{s_i} = \underline{e_1}\,\underline{e_2} \ldots \underline{e_i}$. Notice that for each $\underline{s_i}$ there are precisely $2^i - 1$ idempotents greater than or equal to $\underline{s_i}$, so that each $\underline{s_i}$ lies in a distinct $\sim_{S,1}$-class. Thus I is finite. The converse is immediate by Corollary 2.1. □

Given the disappointing nature of Proposition 4.5 we focus attention on a different construction, which yields useful results. The basic definitions and results are taken from [3].

A semigroup with zero S is a *0-direct union* or *orthogonal sum* of the subsemigroups S_i $(i \in A)$ with zero, if the following hold:

(1) $S_i \neq \{0\}$ for each $i \in A$;
(2) $S = \bigcup_{i \in A} S_i$;
(3) $S_i \cap S_j = S_i S_j = \{0\}$ for each $i \neq j$.

We denote S as $\bigsqcup^0_{i \in A} S_i$. The family $\mathcal{S} = \{S_i : i \in A\}$ is called a *0-direct decomposition* of S, and the S_i are called the *summands* of S. Note that each summand of S forms an ideal of S. If $\mathcal{S}$ and $\mathcal{S}'$ are a pair of 0-direct decompositions of S, then we say that $\mathcal{S}$ is *greater than* $\mathcal{S}'$ if each member of $\mathcal{S}$ is a subsemigroup of some member of $\mathcal{S}'$. We say that S is *0-directly indecomposable* if $\{S\}$ is the unique 0-direct decomposition of S.

Example 4.6 Let $\mathcal{B} = \mathcal{B}^0[G; I]$ be a Brandt semigroup, and consider the group with zero $B_i^0 = \{(i, g, i) : g \in G\} \cup \{0\}$ for each $i \in I$. Then $B_i^0 B_j^0 = \{0\}$ if $i \neq j$, and so $\bigcup_{i \in I} B_i^0$ forms a 0-direct union of the subsemigroups B_i^0. Note that if $|I| > 1$ then $\bigcup_{i \in I} B_i^0$ forms a proper subsemigroup of $\mathcal{B}$, since it does not contain the element (i, g, j) for any $i \neq j$ and $g \in G$.

A subset T of a semigroup S is *consistent* if, for $x, y \in S$, $xy \in T$ implies that $x, y \in T$. A subset T of a semigroup with zero is *0-consistent* if $T \backslash \{0\}$ is consistent. The integral connection between 0-consistency and 0-direct decompositions is that a semigroup with zero S is 0-directly indecomposable if and only if S has no proper 0-consistent ideals [3, Lemma 4]. Consequently, every completely 0-simple semigroup is 0-directly indecomposable.

The central result of [3] was proving that that every semigroup with zero has a greatest 0-direct decomposition, and that the summands of such a decomposition are precisely the 0-directly indecomposable ideals. The importance of the existence of a greatest 0-direct decomposition for $\aleph_0$-categoricity is highlighted in the following proposition.

Proposition 4.7 *Let S be a semigroup with zero and let $\mathcal{S}=\{S_i : i \in A\}$ be the greatest 0-direct decomposition of S. Let $\pi : A \to A$ be a bijection and $\phi_i : S_i \to S_{i\pi}$ an isomorphism for each $i \in A$. Then the map $\phi : S \to S$ given by*

$$s_i\phi = s_i\phi_i \quad (s_i \in S_i)$$

is an automorphism of S, denoted $\phi = \bigsqcup_{i\in A}^0 \phi_i$. Moreover, every automorphism of S can be constructed in this way.

Proof Let ϕ be constructed as in the hypothesis of the proposition. Since $0\phi_i = 0$ for each $i \in A$ the map is well-defined, and it is clearly bijective. Let $a \in S_i$ and $b \in S_j$. If $i = j$ then

$$(ab)\phi = (ab)\phi_i = (a\phi_i)(b\phi_i) = (a\phi)(b\phi),$$

and if $i \neq j$ then

$$(ab)\phi = 0\phi = 0 = (a\phi_i)(b\phi_j) = (a\phi)(b\phi).$$

Hence ϕ is an isomorphism.

Conversely, if ϕ' is an automorphism of S, then

$$\mathcal{S}\phi' = \{S_i\phi' : i \in A\}$$

is clearly a 0-direct decomposition of S. For each summand S_i there exists $k \in A$ such that $S_i \subseteq S_k\phi'$ since $\mathcal{S}$ is the greatest 0-direct decomposition. If $S_i \subseteq S_k\phi' \cap S_{k'}\phi'$ then $S_i = \{0\}$ as $\mathcal{S}\phi'$ is a 0-direct decomposition of S, a contradiction. Hence the element k is unique. On the other hand, if $S_i, S_j \subseteq S_k\phi'$, then $S_i\phi'^{-1}, S_j\phi'^{-1} \subseteq S_k$, and so as $\{S_i\phi'^{-1} : i \in A\}$ is also a 0-direct decomposition of S, we have that $i = j$ since S_k is 0-direct indecomposable. Hence there exists a bijection π' of A such that $S_i\phi' = S_{i\pi'}$ for each $i \in A$ as required. □

Theorem 4.8 *Let S be a semigroup with zero and let $\mathcal{S}=\{S_i : i \in A\}$ be the greatest 0-direct decomposition of S. Then S is $\aleph_0$-categorical if and only if each S_i is $\aleph_0$-categorical and $\mathcal{S}$ is finite, up to isomorphism.*

Proof It follows immediately from Proposition 4.7 that $\{(S_i, x_i) : i \in A\}$ forms a system of 1-pivoted p.r.c subsemigroups of S for any $x_i \in S_i^* = S_i\backslash\{0\}$. Hence if S is $\aleph_0$-categorical then each S_i is $\aleph_0$-categorical and $\mathcal{S}$ is finite, up to isomorphism, by Proposition 3.3.

For the converse direction, let each summand be $\aleph_0$-categorical and suppose there exists exactly $r \in \mathbb{N}$ summands, up to isomorphism. Let $S_{\rho_1}, \ldots, S_{\rho_r}$ be representatives of the isomorphism types of the summands of S and, for each $\mu \in A$, let ϕ_μ be an isomorphism from S_μ to its unique isomorphic representative in $S_{\rho_1}, \ldots, S_{\rho_r}$. By Proposition 2.11 it suffices to show that the number of n-automorphism types of $S^* = S\backslash\{0\}$ is finite for each $n \in \mathbb{N}$. Let $\overline{a} = (a_1, \ldots, a_n)$ and $\overline{b} = (b_1, \ldots, b_n)$ be a pair of n-tuples of S^* with $a_k \in S_{i_k}$ and $b_k \in S_{j_k}$ for each $1 \leq k \leq n$, say. Impose the condition that a_i, a_j belong to the same summand if and only if b_i, b_j belong to the same summand, for each $1 \leq i, j \leq n$. This is clearly equivalent to the map $\pi : \{i_1, \ldots, i_n\} \to \{j_1, \ldots, j_n\}$ given by $i_k\pi = j_k$ being a bijection, and thus the number of choices for this condition is equal to B_n. Suppose also that $S_{i_k} \cong S_{j_k}$ for each k, noting that this condition has r^n choices. For each $1 \leq s \leq r$, let $a_{s1}, \ldots, a_{sn_s}$ be precisely the entries of $\overline{a}$ which are elements of summands isomorphic to S_{ρ_s}, noting that the same is true of $b_{s1}, \ldots, b_{sn_s}$ by our second condition. Note also that $\{1, \ldots, n\} = \{11, \ldots, 1n_1, 21, \ldots, 2n_2, \ldots, rn_r\}$. We impose a final condition on our pair of n-tuples which forces, for each $1 \leq s \leq r$,

$$(a_{s1}\phi_{i_{s1}}, \ldots, a_{sn_s}\phi_{i_{sn_s}}) \sim_{S_{\rho_s}, n_s} (b_{s1}\phi_{j_{s1}}, \ldots, b_{sn_s}\phi_{j_{sn_s}})$$

via $\psi_s \in \mathrm{Aut}(S_{\rho_s})$, say (where if $n_s = 0$ then we take any automorphism of S_{ρ_s}). By the $\aleph_0$-categoricity of each S_{ρ_s} this condition also has finitely many choices. For each $1 \leq s \leq r$ and $1 \leq t \leq n_s$ we have that $\phi_{i_{st}}\psi_s\phi_{j_{st}}^{-1}$ is an isomorphism from $S_{i_{st}}$ to $S_{j_{st}}$, and is such that

$$a_{st}\phi_{i_{st}}\psi_s\phi_{j_{st}}^{-1} = b_{st}.$$

Let π' be a bijection of A which extends π and which preserves the isomorphism types of the summands, so that $S_i \cong S_{i\pi'}$. Such a bijection exists since each S_{i_k} is isomorphic to S_{j_k}. For each $i \in A\backslash\{i_1, \ldots, i_n\}$, let Ψ_i be an isomorphism from S_i to $S_{i\pi'}$, and we let $\Psi_{i_{st}} = \phi_{i_{st}}\psi_s\phi_{j_{st}}^{-1}$ for each $1 \leq s \leq r$ and $1 \leq t \leq n_s$. Then $\Psi = \bigsqcup^0_{i \in A} \Psi_i$ is an automorphism of S by Proposition 4.7, and is such that $\overline{a}\Psi = \overline{b}$ since Ψ extends each $\phi_{i_{st}}\psi_s\phi_{j_{st}}^{-1}$. Since each of our conditions has only finitely many choices, $(S^*)^n$ has only finitely many n-automorphism types (over S) by Lemma 2.8, and thus S is $\aleph_0$-categorical. $\square$

When studying $\aleph_0$-categorical semigroups with zero, it therefore suffices to examine 0-directly indecomposable semigroups.

We observe that without the condition of S being the greatest 0-direct decomposition of S, the converse direction of Theorem 4.8 need not be true. For example, for each $n \in \mathbb{N}$, let N_n be a null semigroup on n non-zero elements. Then $N = \bigsqcup^0_{i \in \mathbb{N}} N_i$ is a countably infinite null semigroup, and is thus $\aleph_0$-categorical by Example 2.7. However the set of summands of N is not finite, up to isomorphism.

A semigroup S with zero is called *primitive* if each of its non-zero idempotents is primitive. It follows from the work of Hall in [17] that a regular semigroup S is primitive if and only if S is isomorphic to a 0-direct union of completely 0-simple semigroups. Since each completely 0-simple semigroup is 0-directly indecomposable, we obtain the following immediate consequence to Theorem 4.8.

Corollary 4.9 *Let S_i ($i \in A$) be a collection of completely 0-simple semigroups. Then $\bigsqcup^0_{i\in A} S_i$ is $\aleph_0$-categorical if and only if each S_i is $\aleph_0$-categorical and $\{S_i : i \in A\}$ is finite, up to isomorphism.*

A classification of primitive regular $\aleph_0$-categorical semigroups via its completely 0-simple semigroup ideals then follows. In particular, by Proposition 4.1 and Theorem 4.2 we have the following classification of primitive inverse semigroups:

Corollary 4.10 *A primitive inverse semigroup S is $\aleph_0$-categorical if and only if S is isomorphic to $\bigsqcup^0_{i\in A} \mathcal{B}^0[G_i; I_i]$ such that each G_i is $\aleph_0$-categorical, $\{G_i : i \in A\}$ is finite up to isomorphism, and $\{|I_i| : i \in A\}$ is finite.*

5 Building $\aleph_0$-categorical semigroups: Semidirect products and $\mathcal{P}$-semigroups

Given that $\aleph_0$-categoricity has been shown by Grzegorczyk to be inherited by finite direct products [16], the next natural question is to assess semidirect products; in this section we do so in the case of a semigroup acting on a finite semigroup. Our work requires the following variant of $\aleph_0$-categoricity, and the subsequent pair of lemmas:

Definition 5.1 Given a semigroup S and a collection $\mathcal{A} = \{S_i : i \in A\}$ of subsets of S, we let $\mathrm{Aut}(S; \mathcal{A})$ denote the group of automorphisms of S which fix each S_i ($i \in A$) setwise. We call S $\aleph_0$-*categorical over* $\mathcal{A}$ if $\mathrm{Aut}(S; \mathcal{A})$ has finitely many orbits on its action on S^n for each $n \in \mathbb{N}$. We let $\sim_{S,\mathcal{A},n}$ denote the resulting equivalence relation on S^n.

With notation as above, Definition 5.1 is equivalent to the structure consisting of the semigroup S together with a collection of unary relations corresponding to the subsets S_i ($i \in I$), being $\aleph_0$-categorical. Moreover, if $\overline{X} = (x_1, \ldots, x_t)$ is a tuple of elements of S, then the condition that S is $\aleph_0$-categorical over $\overline{X}$ is equivalent to S being $\aleph_0$-categorical over $\{\{x_1\}, \ldots, \{x_t\}\}$.

Lemma 5.2 *Let S be a semigroup with a system of t-pivoted p.r.c. subsets $\{(S_i, \overline{X}_i) : i \in I\}$. Then S is $\aleph_0$-categorical over $\mathcal{A} = \{S_i : i \in I\}$ if and only if S is $\aleph_0$-categorical and $\mathcal{A}$ is finite.*

Proof If S is $\aleph_0$-categorical over $\mathcal{A}$, then trivially S is $\aleph_0$-categorical. Suppose $i, j \in I$ are such that $\overline{X}_i \sim_{S,\mathcal{A},t} \overline{X}_j$, via $\phi \in \mathrm{Aut}(S; \mathcal{A})$, say. Then $S_i\phi = S_i$, while $S_i\phi = S_j$ since $\{(S_i, \overline{X}_i) : i \in I\}$ is a system of t-pivoted p.r.c. subsets of S. Hence $S_i = S_j$, and so the cardinality of $\mathcal{A}$ is bound by the number of t-automorphism types over $\mathcal{A}$.

Conversely, suppose S is $\aleph_0$-categorical with $\mathcal{A}$ finite, say $\mathcal{A} = \{S_1, \ldots, S_r\}$. Let $\overline{a} = (a_1, \ldots, a_n)$ and $\overline{b} = (b_1, \ldots, b_n)$ be a pair of n-tuples of S such that $(\overline{a}, \overline{X}_1, \ldots, \overline{X}_r) \sim_{S,n+rt} (\overline{b}, \overline{X}_1, \ldots, \overline{X}_r)$, via $\psi \in \mathrm{Aut}(S)$, say. Then as each pivot is fixed by ψ, the sets S_i are fixed setwise by ψ, so that $\psi \in \mathrm{Aut}(S; \mathcal{A})$. Hence as $\overline{a}\psi = \overline{b}$ we have that

$$|S^n/\sim_{S,\mathcal{A},n}| \leq |S^{n+rt}/\sim_{S,n+rt}| < \aleph_0$$

and so S is $\aleph_0$-categorical over $\mathcal{A}$. □

A simple adaptation of the proof of the lemma above also gives:

Lemma 5.3 *Let S be a semigroup, let $t, r \in \mathbb{N}$, and for each $k \in \{1, \ldots, r\}$ let $\overline{X}_k \in S^t$. Suppose also that S_k is an $\overline{X}_k$-pivoted relatively characteristic subset of S for $1 \leq k \leq r$. Then S is $\aleph_0$-categorical if and only if S is $\aleph_0$-categorical over $\{S_1, \ldots, S_r\}$.*

Now suppose S is a semigroup acted on (on the left) by a monoid T via endomorphisms. That is, we have a map $T \times S \to S$ denoted by $(t, s) \mapsto t \cdot s$, such that for all $t, t' \in T$ and $s, s' \in S$ we have $tt' \cdot s = t \cdot (t' \cdot s)$, $1 \cdot s = s$ and $t \cdot (ss') = (t \cdot s)(t \cdot s')$. We may then construct a semigroup on the set $S \times T$ with binary operation $(s, t)(s', t') = (s(t \cdot s'), tt')$. The resulting semigroup is denoted by $S \rtimes T$, and is called a *semidirect product* of S by T. We refer the reader to [40] for an introduction to monoid actions.

Given a semidirect product $S \rtimes T$, we define a relation κ on T by

$$t\,\kappa\,t' \Leftrightarrow s(t \cdot s') = s(t' \cdot s') \ (\forall s, s' \in S). \tag{5.1}$$

Then κ is clearly an equivalence relation on T, and if S is finite then T/κ is finite. If S is a monoid and $t \cdot 1_S = 1_S$ for all $t \in T$, then we say that T acts *monoidally*; note in this case $S \rtimes T$ is a monoid, and the definition of κ simplifies to

$$t\,\kappa\,t' \Leftrightarrow t \cdot s = t' \cdot s \ (\forall s \in S).$$

Theorem 5.4 *Let $M = S \rtimes T$ be a semidirect product of S and T, where S is finite. If T is $\aleph_0$-categorical over T/κ, then M is $\aleph_0$-categorical.*

The converse holds if S is a monoid with trivial group of units and T acts monoidally, or if S is a semilattice.

Proof Suppose that T is $\aleph_0$-categorical over T/κ. Let $\overline{a} = ((s_1, t_1), \ldots, (s_n, t_n))$ and $\overline{b} = ((s_1', t_1'), \ldots, (s_n', t_n'))$ be n-tuples of M under the conditions that

(1) $s_k = s_k'$ for each $1 \leq k \leq n$,
(2) $(t_1, \ldots, t_n) \sim_{T,T/\kappa,n} (t_1', \ldots, t_n')$ via $\theta \in \mathrm{Aut}(T; T/\kappa)$, say.

Note that the first condition has $|S|^n$ choices. We claim that the bijection $\phi : M \to M$ given by $(s, t)\phi = (s, t\theta)$ is an automorphism of M. Given $(s, t), (s', t') \in M$,

$$\begin{aligned}((s, t)(s', t'))\phi &= (s(t \cdot s'), tt')\phi = (s(t \cdot s'), (tt')\theta) = (s(t\theta \cdot s'), (tt')\theta)\\ &= (s(t\theta \cdot s'), (t\theta)(t'\theta)) = (s, t\theta)(s', t'\theta) = (s, t)\phi(s', t')\phi,\end{aligned}$$

where the third equality is due to $t\,\kappa\,t\theta$, so in particular $s(t \cdot s') = s(t\theta \cdot s')$. Hence ϕ is indeed an automorphism of M. Moreover, for each $1 \leq k \leq n$,

$$(s_k, t_k)\phi = (s_k, t_k\theta) = (s_k', t_k')$$

and so $\overline{a}\phi = \overline{b}$. We therefore have that

$$|M^n/\sim_{M,n}| \le |S|^n \cdot |T^n/\sim_{T,T/\kappa,n}| < \aleph_0$$

since S is finite and T is $\aleph_0$-categorical over T/κ. Hence M is $\aleph_0$-categorical by the RNT.

Conversely, suppose that M is $\aleph_0$-categorical. Enumerate the elements of S as $\{s_1, \ldots, s_r\}$. Let $s = 1_S$ if S is a monoid and let $s = 0$ where 0 is the least idempotent of S if S is a semilattice. Let $\overline{t} = (t_1, \ldots, t_n)$ and $\overline{u} = (u_1, \ldots, u_n)$ be n-tuples of T under the conditions that

(1) $t_k \,\kappa\, u_k$ for each $1 \le k \le n$,
(2) $\big((s_1, 1), \ldots, (s_r, 1), (s, t_1), \ldots, (s, t_n)\big)$
$\sim_{M,n+r} \big((s_1, 1), \ldots, (s_r, 1), (s, u_1), \ldots, (s, u_n)\big)$ via $\theta \in \mathrm{Aut}(M)$, say.

Recalling that s is fixed, for any $t \in T$ we define $t\phi$ to be the second coordinate of $(s, t)\theta$. We claim that $\phi \in \mathrm{Aut}(T)$ and preserves T/κ.

Case (i): S a monoid with trivial group of units, so $s = 1_S$. We first show that $(1_S, t)\theta = (1_S, t\phi)$. By definition, we have $(1_S, t)\theta = (s', t\phi)$ for some $s' \in S$. Choose $(b, w) \in S \rtimes T$ such that $(b, w)\theta = (1_S, t\phi)$. Then θ fixes $(s', 1)$ by condition (2), so that

$$(1_S, t)\theta = (s', t\phi) = (s', 1)(1_S, t\phi) = (s', 1)\theta(b, w)\theta = ((s', 1)(b, w))\theta = (s'b, w)\theta.$$

Hence $1_S = s'b$, giving $s' = 1_S$ as H_{1_S} is trivial and S is finite (giving that an element with a left inverse lies in H_{1_S}).

We have thus shown that $(1_S, t)\theta = (1_S, t\phi)$, whence it follows that for any $u \in S, t \in T$ we have

$$(u, t)\theta = \big((u, 1)(1_S, t)\big)\theta = (u, 1)(1_S, t\phi) = (u, t\phi).$$

It is now easy to see that $\phi \in \mathrm{Aut}(T)$, since T acts monoidally.

Case (ii): S a semilattice, so $s = 0$. We first show that $(0, t)\theta = (0, t\phi)$. To see this, let $(0, t)\theta = (0', t\phi)$; making use of (2) we have

$$(0', t\phi) = (0, t)\theta = \big((0, 1)(0, t)\big)\theta = (0, 1)(0', t\phi) = (0, t\phi)$$

giving $0' = 0$. It is now easy to see that ϕ yields an automorphism of T.

Let $e \in S$ and $t \in T$ and suppose that $(e, t)\theta = (e', t')$. Then

$$(0, t') = (0, 1)(e', t') = (0, 1)\theta(e, t)\theta = \big((0, 1)(e, t)\big)\theta = (0, t)\theta = (0, t\phi),$$

so that $t' = t\phi$. Let $u \in S$ with $u > 0$. We may then suppose for induction that for all $v \in S$ with $u > v$ and for all $t \in T$ we have $(v, t)\theta = (v, t\phi)$. Then with $(u, t)\theta = (u', t\phi)$ we have

$$(u', t\phi) = (u, t)\theta = \big((u, 1)(u, t)\big)\theta = (u, 1)(u', t\phi) = (uu', t\phi),$$

so that $u' = uu'$ and $u' \leq u$. If $u' < u$ we are led to the contradiction that $(u', t)\theta = (u', t\phi) = (u, t)\theta$. Thus $u' = u$ and we deduce that for any $w \in S, t \in T$ we have $(w, t)\theta = (w, t\phi)$.

In each case, for any $u, u' \in S$ and $t \in T$, by applying θ to the product $(u, t)(u', 1)$ we immediately see that $t \,\kappa\, t\phi$. Moreover, as $(s, t_i)\theta = (s, u_i)$ we have $t_i\phi = u_i$ for $1 \leq i \leq n$. Thus T is $\aleph_0$-categorical over T/κ. □

Open Problem 5.5 Can we weaken the conditions on S in the converse to Theorem 5.4?

Example 5.6 Let T be a semigroup acting trivially on a finite semigroup S, so that $t \cdot s = s$ for each $s \in S, t \in T$. It follows that κ is the universal relation. Hence $S \rtimes T$ is $\aleph_0$-categorical if T is $\aleph_0$-categorical over $T/\kappa = \{T\}$, which is clearly equivalent to T being $\aleph_0$-categorical. Note that $S \rtimes T$ is simply the direct product of S and T, and so we recover Grzegorczyk's result [16].

Example 5.7 Let $L = \{x_1, \ldots, x_r\}$ be a finite left zero band and $S = \bigsqcup^0_{i \in I} S_i$ an $\aleph_0$-categorical 0-direct union of 0-directly indecomposable S_i. Then as $L^0 = L \cup \{0\}$ is 0-directly indecomposable and $\aleph_0$-categorical, it follows from Theorem 4.8 that $S' = S \sqcup^0 L^0$ is also $\aleph_0$-categorical. Let S' act on its ideal L^0 by left multiplication, so that $t \cdot s = ts$ for each $t \in S'$ and $s \in L^0$. Then this is an action by endomorphisms as

$$t \cdot (s_1 s_2) = t s_1 s_2 = \begin{cases} t & \text{if } t \in L^0 \\ 0 & \text{else} \end{cases} = (ts_1)(ts_2).$$

We aim to show that $L^0 \rtimes S'$ is $\aleph_0$-categorical. Notice that $t \,\kappa\, t'$ if and only if $sts' = st's'$ for all $s, s' \in L^0$. However,

$$sts' = \begin{cases} s & \text{if } t \in L, \\ 0 & \text{else} \end{cases}$$

and it follows that the κ-classes are L and $S' \backslash L$. Since any automorphism of S' which fixes L setwise clearly fixes $S' \backslash L$ setwise, we have that S' is $\aleph_0$-categorical over S'/κ if and only if S' is $\aleph_0$-categorical over $\{L\}$. From Lemma 2.6, we have that S' is $\aleph_0$-categorical over $(x_1, \ldots, x_r)$, hence over $\{L\}$, and so over S'/κ. Hence $L^0 \rtimes S'$ is $\aleph_0$-categorical by Theorem 5.4.

Our final example comes from studying the semidirect product of a group and a semilattice. Such semigroups are examples of E-unitary inverse semigroups, a class that plays a central role in the study of inverse semigroups. A semigroup S is *E-unitary* if whenever $e, es \in E(S)$ then $s \in E(S)$. In the case of inverse semigroups, this condition is equivalent to $\mathcal{R} \cap \sigma = \iota$ (or, indeed, to $\mathcal{L} \cap \sigma = \iota$), where σ is the the congruence defined in Example 3.5 (2), and ι is the equality relation. This equivalent condition is often referred to as that of being *proper*. McAlister [22,23] showed that every inverse semigroup has an E-unitary cover (a pre-image via an idempotent separating morphism) and characterised the structure of E-unitary semigroups via what are known as $\mathcal{P}$-semigroups. The construction of a $\mathcal{P}$-semigroup, which can be found in [19, Chapter 5], is very close to that of a semidirect product of a semilattice by a group. Indeed, a semidirect product of a semilattice by a group is E-unitary, and every $\mathcal{P}$-semigroup embeds into such a semidirect product [26]. However, a free inverse semigroup provides an example of a $\mathcal{P}$-semigroup which is not a semidirect product of a semilattice by a group.

Let $\mathcal{X}$ be a partially ordered set with order $\leq$, and let $\mathcal{Y}$ be an order ideal of $\mathcal{X}$ which forms a semilattice under $\leq$. Let G be a group which acts on $\mathcal{X}$ by order automorphisms, and suppose in addition that

(1) $G\mathcal{Y} = \mathcal{X}$ and
(2) $g\mathcal{Y} \cap \mathcal{Y} \neq \emptyset$ for all $g \in G$.

Then the triple $(G, \mathcal{X}, \mathcal{Y})$ is called a *McAlister triple*. We may then take

$$\mathcal{P} = \mathcal{P}(G, \mathcal{X}, \mathcal{Y}) = \{(A, g) \in \mathcal{Y} \times G : g^{-1}A \in \mathcal{Y}\}$$

and define an operation on $\mathcal{P}$ by the rule

$$(A, g)(B, h) = (A \wedge gB, gh).$$

Theorem 5.8 [23] *Let* $(G, \mathcal{X}, \mathcal{Y})$ *be a* McAlister triple. *Then* $\mathcal{P} = \mathcal{P}(G, \mathcal{X}, \mathcal{Y})$ *is an* E*-unitary inverse semigroup with semilattice of idempotents* $E(\mathcal{P}) = \mathcal{Y} \times \{1\}$ *isomorphic to* $\mathcal{Y}$. *Moreover, any* E*-unitary inverse semigroup* S *is isomorphic to some* $\mathcal{P}(G, \mathcal{X}, \mathcal{Y})$ *where* $G = S/\sigma$ *and* $\mathcal{Y} = E(S)$.

The semigroup $\mathcal{P}(G, \mathcal{X}, \mathcal{Y})$ is often referred to as a $\mathcal{P}$*-semigroup*. Notice that if $\mathcal{X} = \mathcal{Y}$ then $\mathcal{P}(G, \mathcal{X}, \mathcal{Y}) = \mathcal{Y} \rtimes G$. One can show that for any $(A, g), (B, h) \in \mathcal{P}(G, \mathcal{X}, \mathcal{Y})$ we have

$$(A, g)\,\mathcal{R}\,(B, h) \Leftrightarrow A = B \text{ and } (A, g)\,\mathcal{L}\,(B, h) \Leftrightarrow g^{-1}A = h^{-1}B.$$

For any $\mathcal{P}$-semigroup $\mathcal{P} = \mathcal{P}(G, \mathcal{X}, \mathcal{Y})$ we have $\mathcal{P}/\sigma \cong G$, and so the $\aleph_0$-categoricity of $\mathcal{P}$ passes to G by Corollary 3.11 (2). Our aim is therefore to consider when the converse holds, or rather, what conditions on G force $\mathcal{P}$ to be $\aleph_0$-categorical? We require McAlister's [23] description of morphisms between $\mathcal{P}$-semigroups, which simplifies to automorphisms as follows.

Proposition 5.9 *Let* $\mathcal{P} = \mathcal{P}(G, \mathcal{X}, \mathcal{Y})$ *be a* $\mathcal{P}$*-semigroup. Let* $\theta \in \mathrm{Aut}(G)$ *and* $\psi : \mathcal{X} \to \mathcal{X}$ *be an order-automorphism such that* $\psi|_{\mathcal{Y}} \in \mathrm{Aut}(\mathcal{Y})$. *Suppose also that, for all* $g \in G$ *and* $A \in \mathcal{X}$,

$$(gA)\psi = (g\theta)(A\psi). \tag{5.2}$$

Then the map $\phi : \mathcal{P} \to \mathcal{P}$ *given by* $(A, g)\phi = (A\psi, g\theta)$ *is an automorphism, denoted* $\phi = (\psi; \theta)$. *Conversely, every automorphism of* $\mathcal{P}$ *is of this type.*

Notice that if ψ is the identity on $\mathcal{X}$, then for (5.2) to hold the automorphism θ of G must preserve the group action, that is,

$$gA = (g\theta)A \quad (\forall g \in G)(\forall A \in \mathcal{X}). \tag{5.3}$$

On the other hand, if θ is the identity on G, then for (5.2) to hold ψ must be a G-act morphism, in addition to being an order-automorphism, that is,

$$(gA)\psi = g(A\psi). \tag{5.4}$$

We first consider the case of finite $\mathcal{Y}$. We recall from Sect. 3 that we denote by μ the largest idempotent separating congruence on an inverse semigroup S, and that μ is also precisely the largest congruence contained in $\mathcal{H}$. From the formula for μ on [19, p.160], it is clear that μ has finitely many classes if and only if $E(S)$ is finite, so that if $\mathcal{P} = \mathcal{P}(G, \mathcal{X}, \mathcal{Y})$, then $\mathcal{P}/\mu$ is finite if and only if $\mathcal{Y}$ is finite.

For each $A \in \mathcal{Y}$ let $T_A \subseteq G$ be defined by

$$T_A = \{g \in G : g^{-1}A \in \mathcal{Y}\},$$

noting that $G = \bigcup_{A\in\mathcal{Y}} T_A$ since $g\mathcal{Y} \cap \mathcal{Y} \neq \emptyset$ for each $g \in G$. We also define a relation $\sim_A$ on T_A by:

$$g \sim_A h \Leftrightarrow g^{-1}U = h^{-1}U \text{ for all } U \leq A.$$

Clearly each $\sim_A$ is an equivalence.

The following lemma is immediate from [19, p.217].

Lemma 5.10 *Let $\mathcal{P} = \mathcal{P}(G, \mathcal{X}, \mathcal{Y})$ be a $\mathcal{P}$-semigroup. Then*

$$(A, g)\, \mu\, (B, h) \Leftrightarrow A = B \text{ and } g \sim_A h.$$

Remark 5.11 The finiteness of $\mathcal{Y}$ in $\mathcal{P} = \mathcal{P}(G, \mathcal{X}, \mathcal{Y})$ does not imply that $\mathcal{X}$ is finite.

To see this, consider the semilattice $\mathcal{X}$ on $\{X_i : i \in \mathbb{N}^0\} \cup \{0\}$ where 0 is a least element and the remaining elements are atoms. Let G be the free group on $\{g_i : i \in \mathbb{N}\}$. Each g_i determines an order-automorphism of $\mathcal{X}$ by setting $g_iX_0 = X_i$, $g_iX_i = X_0$ and $g_iX_j = X_j$ for all $j \neq 0, i$. Since G is free on $\{g_i : i \in \mathbb{N}\}$, we may lift the action of the generators to an action of G on $\mathcal{X}$ by order-automorphisms. Setting $\mathcal{Y} = \{X_0, 0\}$ we see that $(G, \mathcal{X}, \mathcal{Y})$ is a McAlister triple.

Theorem 5.12 *Let $\mathcal{P} = \mathcal{P}(G, \mathcal{X}, \mathcal{Y})$ be a $\mathcal{P}$-semigroup such that $\mathcal{Y}$ is finite. Then $\mathcal{P}$ is $\aleph_0$-categorical if and only if G is $\aleph_0$-categorical over $\bigcup_{A\in\mathcal{Y}} T_A/\sim_A$.*

Proof Let $\mathcal{A} = \bigcup_{A\in\mathcal{Y}} T_A/\sim_A$. Let $\mathcal{Y} = \{Y_1, \ldots, Y_k\}$ and for each $Y_i \in \mathcal{Y}$ pick a set of representatives

$$(Y_i, g_1^i), \ldots, (Y_i, g_{n(i)}^i)$$

of the μ-classes sitting in the $\mathcal{R}$-class of $(Y_i, 1)$, so that $g_1^i, \ldots, g_{n(i)}^i$ are representatives of the $\sim_{Y_i}$-classes.

($\Rightarrow$) Let $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ be a pair of n-tuples of G. Then for any j with $1 \leq j \leq n$, as $a_j^{-1}\mathcal{Y} \cap \mathcal{Y} \neq \emptyset$ there exists $A_j \in \mathcal{Y}$ such that $(A_j, a_j) \in \mathcal{P}$; similarly form $(B_j, b_j) \in \mathcal{P}$. Impose the condition that

$$\big((Y_1, g_1^1), \ldots, (Y_1, g_{n(1)}^1), \ldots, (Y_k, g_1^k), \ldots, (Y_k, g_{n(k)}^k), (A_1, a_1), \ldots\ldots, (A_n, a_n)\big)$$

and

$$\big((Y_1, g_1^1), \ldots, (Y_1, g_{n(1)}^1), \ldots, (Y_k, g_1^k), \ldots, (Y_k, g_{n(k)}^k), (B_1, b_1), \ldots, (B_n, b_n)\big)$$

are related by $\sim_{\mathcal{P},t+n}$ where $t = |\mathcal{P}/\mu|$, via $\phi = (\psi; \theta) \in \text{Aut}(\mathcal{P})$, say. Clearly each μ-class is fixed by ϕ and in particular each $A \in \mathcal{Y}$ is fixed by ψ. Suppose that $g \in T_A$. Then

$$g^{-1}A = (g^{-1}A)\psi = (g^{-1}\theta)(A\psi) = (g^{-1}\theta)A = (g\theta)^{-1}A$$

so that $g\theta \in T_A$. Moreover, for any $U \leq A$ we have $g^{-1}U \in \mathcal{Y}$ and

$$g^{-1}U = (g^{-1}U)\psi = (g\theta)^{-1}(U\psi) = (g\theta)^{-1}U,$$

so that $g \sim_A g\theta$. Thus $\theta \in \text{Aut}(G)$ and θ preserves $\mathcal{A}$. Clearly $a_i\theta = b_i$ for $1 \leq i \leq n$. Hence G is $\aleph_0$-categorical over $\mathcal{A}$.

($\Leftarrow$) Since $\mathcal{Y}$ is finite and G is $\aleph_0$-categorical over $\mathcal{A}$, from any infinite list of n-tuples of elements of $\mathcal{P}$ we may pick out a pair $\overline{a} = ((A_1, g_1), \ldots, (A_n, g_n))$ and $\overline{b} = ((A_1, h_1), \ldots, (A_n, h_n))$ such that $(g_1, \ldots, g_n) \sim_{G,\mathcal{A},n} (h_1, \ldots, h_n)$, via some $\theta \in \text{Aut}(G; \mathcal{A})$. Define $\phi : \mathcal{P} \to \mathcal{P}$ by $(A, g)\phi = (A, g\theta)$. Let $g \in G$ and $A, B \in \mathcal{Y}$ with $g^{-1}A \in \mathcal{Y}$. Since $g \in T_A$ and $A \wedge gB$ exists and is less than A, we have $g^{-1}(A \wedge gB) = (g\theta)^{-1}(A \wedge gB)$. Also, $A \wedge gB = gg^{-1}(A \wedge gB) = g(g^{-1}A \wedge B) = g((g\theta)^{-1}A \wedge B)$. It follows that $A \wedge gB = A \wedge (g\theta)B$ so that, consequently, ϕ is an isomorphism. Hence $\mathcal{P}$ is $\aleph_0$-categorical. □

Given a $\mathcal{P}$-semigroup $\mathcal{P} = \mathcal{P}(G, \mathcal{X}, \mathcal{Y})$, we define a relation ν on G by

$$g \,\nu\, h \Leftrightarrow gA = hA \ (\forall A \in \mathcal{X}). \tag{5.5}$$

Then ν is clearly an equivalence relation, and if $g \,\nu\, h$ then $g \sim_A h$ for any $A \in \mathcal{Y}$, with $g \in T_A$ if and only if $h \in T_A$. Hence if $g \in G$ then $g\nu$ is a subset of an element of $T_A/\sim_A$ for some $A \in \mathcal{Y}$. It follows that if G is $\aleph_0$-categorical over G/ν, then G is $\aleph_0$-categorical over $\mathcal{A}$. If $\mathcal{X}$ is finite, then there are only finitely many ν-classes. Lemma 5.10 and Theorem 5.12 thus prove the reverse direction to the following result:

Theorem 5.13 *Let $\mathcal{P} = \mathcal{P}(G, \mathcal{X}, \mathcal{Y})$ be a $\mathcal{P}$-semigroup such that $\mathcal{X}$ is finite. Then $\mathcal{P}$ is $\aleph_0$-categorical if and only if G is $\aleph_0$-categorical over G/ν.*

Proof Suppose $\mathcal{P}$ is $\aleph_0$-categorical and $\mathcal{X} = \{X_1, \ldots, X_r\}$ is finite. For each $i \in \{1, \ldots, r\}$ choose and fix $g_i \in G$ and $Y_i \in \mathcal{Y}$ such that $X_i = g_i Y_i$. Let $\overline{g} = (a_1, \ldots, a_n)$ and $\overline{h} = (b_1, \ldots, b_n)$ be a pair of n-tuples of G. For each $1 \leq j \leq n$ and $1 \leq i \leq r$, form $(A_j, a_j), (B_j, b_j), (C_i, g_i) \in \mathcal{P}$ for some $A_j, B_j, C_i \in \mathcal{Y}$. Impose the conditions that

$$\big((A_1, a_1), \ldots, (A_n, a_n), (C_1, g_1), \ldots, (C_r, g_r), (Y_1, 1), \ldots, (Y_r, 1)\big)$$

and

$$\big((B_1, b_1), \ldots, (B_n, b_n), (C_1, g_1), \ldots, (C_r, g_r), (Y_1, 1), \ldots, (Y_r, 1)\big)$$

are related by $\sim_{\mathcal{P},n+2r}$, via $\phi = (\psi; \theta) \in \mathrm{Aut}(\mathcal{P})$. Then

$$X_i \psi = (g_i Y_i)\psi = (g_i\theta)(Y_i \psi) = g_i Y_i = X_i$$

for each i, and so $\psi = I_{\mathcal{X}}$. For any $g \in G$ and $X_i \in \mathcal{X}$ we then have

$$gX_i = (gX_i)\psi = (g\theta)(X_i \psi) = (g\theta)X_i,$$

so that $g \, \nu \, g\theta$. Thus $\theta \in \mathrm{Aut}(G)$ preserves G/ν, and is such that $a_j\theta = b_j$ for $1 \leq j \leq n$. Hence G is $\aleph_0$-categorical over G/ν. □

We handle the corresponding case where G is finite by introducing a new structure. In the definition of a $\mathcal{P}$-semigroup $\mathcal{P} = \mathcal{P}(G, \mathcal{X}, \mathcal{Y})$ we have that G acts on the poset $\mathcal{X}$ by order automorphisms. Thus we may refer to $\mathcal{X}$ as a *G-poset* [11] (the partial order on G being equality). We consider the structure obtained by augmenting the signature for G-posets with a unary relation $\mathcal{Z}$ and, suppressing $\leq$ for convenience, refer to the new structure as the *augmented G-poset* $(\mathcal{X}, \mathcal{Z})$. Thus $(\mathcal{X}, \mathcal{Z})$ has universe $\mathcal{X}$ and signature $(\leq, \mathcal{Z}, \lambda_g : g \in G)$ where λ_g is the action of g on $\mathcal{X}$. An automorphism ψ of $(\mathcal{X}, \mathcal{Z})$ must therefore be a G-act isomorphism (that is, $(gX)\psi = g(X\psi)$ for all $g \in G$, $X \in \mathcal{X}$), in addition to being an order automorphism of $\mathcal{X}$ fixing $\mathcal{Z}$ setwise.

Theorem 5.14 *Let $\mathcal{P} = \mathcal{P}(G, \mathcal{X}, \mathcal{Y})$ be a $\mathcal{P}$-semigroup such that G is finite. Then $\mathcal{P}$ is $\aleph_0$-categorical if and only if the augmented G-poset $(\mathcal{X}, \mathcal{Y})$ is $\aleph_0$-categorical.*

Proof Let $G = \{g_1, \ldots, g_m\}$ and pick $D_j \in \mathcal{Y}$ with $(D_j, g_j) \in \mathcal{P}$, for $1 \leq j \leq m$.

Suppose first that $\mathcal{P}$ is $\aleph_0$-categorical. For any $n \in \mathbb{N}$ and infinite sequence of n-tuples of $\mathcal{X}$, we can find $h_1, \ldots, h_n \in G$ and a subsequence in which every n-tuple can be written as $(X_1, \ldots, X_n)$, where $X_i \in h_i\mathcal{Y}$ for all $1 \leq i \leq n$. From the $\aleph_0$-categoricity of $\mathcal{P}$ we can find distinct elements $(h_1 Y_1, \ldots, h_n Y_n)$ and $(h_1 Z_1, \ldots, h_n Z_n)$ of our sequence (where $Y_i, Z_i \in \mathcal{Y}$ for $1 \leq i \leq n$) such that

$$\begin{aligned}&\big((D_1, g_1), \ldots, (D_m, g_m), (Y_1, 1), \ldots, (Y_n, 1)\big)\\ &\quad \sim_{\mathcal{P},m+n} \big((D_1, g_1), \ldots, (D_m, g_m), (Z_1, 1), \ldots, (Z_n, 1)\big)\end{aligned}$$

via $\phi = (\psi; \theta)$, where clearly $\theta = I_G$. Then for any $g \in G$, $X \in \mathcal{X}$ we have

$$(gA)\psi = (g\theta)(A\psi) = g(A\psi)$$

so that ψ is a G-act isomorphism, in addition to possessing the properties that $\psi \in \text{Aut}(\mathcal{X})$ and $\psi|_{\mathcal{Y}} \in \text{Aut}(\mathcal{Y})$. Thus ψ is an automorphism of the augmented G-act $(\mathcal{X}, \mathcal{Y})$. Moreover, we have

$$(h_i Y_i)\psi = h_i(Y_i\psi) = h_i Z_i$$

so that

$$(h_1 Y_1, \ldots, h_n Y_n) \sim_{(\mathcal{X},\mathcal{Y}),n} (h_1 Z_1, \ldots, h_n Z_n)$$

as required. Thus $(\mathcal{X}, \mathcal{Y})$ is $\aleph_0$-categorical.

Conversely, suppose that $(\mathcal{X}, \mathcal{Y})$ is $\aleph_0$-categorical and we have an infinite sequence of n-tuples of $\mathcal{P}$. Since G is finite and $(\mathcal{X}, \mathcal{Y})$ is $\aleph_0$-categorical we may find a distinct pair $\big((A_1, h_1), \ldots, (A_n, h_n)\big)$ and $\big((B_1, h_1), \ldots, (B_n, h_n)\big)$ such that

$$(A_1, \ldots, A_n) \sim_{(\mathcal{X},\mathcal{Y}),n} (B_1, \ldots, B_n)$$

via ψ. As ψ is a G-act isomorphism, it is immediate that $(\psi; I_G)$ is in $\text{Aut}(\mathcal{P})$ and moreover,

$$\big((A_1, h_1), \ldots, (A_n, h_n)\big) \sim_{\mathcal{P},n} \big((B_1, h_1), \ldots, (B_n, h_n)\big)$$

via $(\psi; I_G)$. Thus $\mathcal{P}$ is $\aleph_0$-categorical as required. □

To deal with the case of a $\mathcal{P}$-semigroup where both the semilattice $\mathcal{Y}$ and group G are infinite, we require a little more sophistication. In the sequel to this article we obtain classes of $\aleph_0$-categorical E-unitary semigroups with infinite semilattice of idempotents, by restricting our attention to those with central idempotents.

Acknowledgements This work forms part of the Ph.D. of the second author at the University of York, funded by EPSRC. The authors would also like to thank the referee for his/her very careful reading of their manuscript, and the very detailed and helpful comments provided, which have resulted in a much improved article.

References

1. Apps, A.B.: On the structure of $\aleph_0$-categorical groups. J. Algebra **81**, 320–339 (1982)
2. Baldwin, J., Rose, B.: $\aleph_0$-categoricity and stability of rings. J. Algebra **45**, 1–16 (1977)
3. Bogdanović, S., Ćirić, M.: Orthogonal sums of semigroups. Isr. J. Math. **90**, 423–428 (1995)
4. Cherlin, G.: On $\aleph_0$-categorical nilrings II. J. Symb. Log. **45**, 291–301 (1980)
5. Clifford, A.H., Preston, G.B.: The Algebraic Theory of Semigroups, vol. 1. American Mathematical Society, Province (1961)

6. Droste, M.: Structure of partially ordered sets with transitive automorphism groups. Mem. Am. Math. Soc. **57**, 334 (1985)
7. Droste, M., Kuske, D., Truss, J.K.: On homogeneous semilattices and their automorphism groups. Order **16**, 31–56 (1999)
8. Engeler, E.: A characterization of theories with isomorphic denumerable models. Am. Math. Soc. Not. **6**, 161 (1959)
9. Evans, D.M.: Examples of $\aleph_0$-categorical structures. In: Kaye, R., MacPherson, H.D. (eds.) Automorphisms of First-Order Structures. Oxford Logic Guides, Oxford (1994)
10. Evans, D.M.: Model Theory of Groups and Automorphism Groups. Cambridge University Press, Cambridge (1997)
11. Fakhruddin, S.M.: On the category of S-posets. Acta Sci. Math. (Szeged) **52**, 85–92 (1988)
12. Flaška, V., Kepka, T.: Commutative zeropotent semigroups. Acta Univ. Carolinae **47**, 3–14 (2006)
13. Givant, S., Halmos, P.: Introduction to Boolean Algebras. Springer, Berlin (2008)
14. Gould, V.: Notes on Restriction semigroups and related structures, http://www-users.york.ac.uk/~varg1/restriction.pdf. Accessed 31 July 2017
15. Grätzer, G.: General Lattice Theory. Birkhäuser Verlag, Boston (1998)
16. Grzegorczyk, A.: Logical uniformity by decomposition and categoricity in $\aleph_0$. Bull. Acad. Pol. Sci. Sér. Sci. Math. Astron. Phys. **16**, 687–692 (1968)
17. Hall, T.E.: On the natural ordering of $\mathcal{J}$-classes and of idempotents in a regular semigroup. Glasgow Math. J. **11**, 167–168 (1970)
18. Hodges, W.: Model Theory. Cambridge University Press, Cambridge (1993)
19. Howie, J.M.: Fundamentals of Semigroup Theory. Oxford University Press, Oxford (1995)
20. Ježek, J., Kepka, T., Němec, P.: Commutative semigroups that are nil of index 2 and have no irreducible elements. Math. Bohem. **133**, 1–7 (2008)
21. Landman, F.: Structures for Semantics. Springer, Amsterdam (1991)
22. McAlister, D.B.: Groups, semilattices and inverse semigroups. Trans. Am. Math. Soc. **192**, 227–244 (1974)
23. McAlister, D.B.: Groups, semilattices and inverse semigroups II. Trans. Am. Math. Soc. **196**, 351–370 (1974)
24. McLean, D.: Idempotent semigroups. Am. Math. Mon. **61**, 110–113 (1954)
25. Morley, M.: Categoricity in power. Trans. Am. Math. Soc. **114**, 514–538 (1965)
26. O'Carroll, L.: Embedding theorems for proper inverse semigroups. J. Algebra **42**, 26–40 (1976)
27. Olin, P.: $\aleph_0$-categoricity of two-sorted structures. Algebra Universalis **2**, 262–269 (1972)
28. Pillay, A.: An Introduction to Stability Theory. Dover Books on Mathematics, New York (2008)
29. Quinn-Gregson, T.: $\aleph_0$-categoricity of semigroups II (Submitted). arXiv:1803.10087
30. Quinn-Gregson, T.: Homogeneity and $\aleph_0$-categoricity of semigroups. PhD Thesis, University of York (2017)
31. Quinn-Gregson, T.: Homogeneous bands. Adv. Math. **328**, 623–660 (2018)
32. Quinn-Gregson, T.: Homogeneity of inverse semigroups. Int. J. Algebra. Comput. **28**, 837–875 (2018)
33. Rosenstein, J.G.: $\aleph_0$-categoricity of linear orderings. Fund. Math. **64**, 1–5 (1969)
34. Rosenstein, J.G.: $\aleph_0$-categoricity of groups. J. Algebra **25**, 435–467 (1973)
35. Rosenstein, J.G.: $\aleph_0$-categoricity is not inherited by factor groups. Algebra Universalis **6**, 93–95 (1976)
36. Rota, G.C.: The number of partitions of a set. Am. Math. Mon. **71**, 498–504 (1964)
37. Ryll-Nardzewski, C.: On the categoricity in power $\leq \aleph_0$. Bull. Acad. Pol. Ser. Sci. Math. Astron. Phys. **7**, 545–548 (1959)
38. Sabbagh, G.: Catégoricité et stabilité: quelques exemples parmi les groupes et anneaux. C.R. Acad. Sci. Paris Sér. A **280**, 603–606 (1975)
39. Schmerl, J.H.: On $\aleph_0$-categoricity of filtered Boolean extensions. Algebra Universalis **8**, 159–161 (1978)
40. Steinberg, B.: A theory of transformation monoids: combinatorics and representation theory. Electron. J. Comb. **17**, 1–56 (2010). (Research Paper 164)
41. Svenonius, L.: $\aleph_0$-categoricity in first-order predicate calculus. Theoria **25**, 82–94 (1959)
42. Waszkiewicz, J., Weglorz, B.: On ω_0-categoricity of powers. Bull. Acad. Pol. Sci. Sk. Sci. Math. Astron. Phys. **17**, 195–199 (1969)